

# Wavefront Sets in Algebraic Quantum Field Theory

RAINER VERCH

Institut für Theoretische Physik,

Universität Göttingen,

Bunsenstr. 9,

D-37073 Göttingen, Germany

e-mail: verch@theorie.physik.uni-goettingen.de

**Abstract:** The investigation of wavefront sets of  $n$ -point distributions in quantum field theory has recently acquired some attention stimulated by results obtained with the help of concepts from microlocal analysis in quantum field theory in curved spacetime. In the present paper, the notion of wavefront set of a distribution is generalized so as to be applicable to states and linear functionals on nets of operator algebras carrying a covariant action of the translation group in arbitrary dimension. In the case where one is given a quantum field theory in the operator algebraic framework, this generalized notion of wavefront set, called “asymptotic correlation spectrum”, is further investigated and several of its properties for physical states are derived. We also investigate the connection between the asymptotic correlation spectrum of a physical state and the wavefront sets of the corresponding Wightman distributions if there is a Wightman field affiliated to the local operator algebras. Finally we present a new result (generalizing known facts) which shows that certain spacetime points must be contained in the singular supports of the  $2n$ -point distributions of a non-trivial Wightman field.

## 1 Introduction

The wavefront set of a distribution (see the next section for a definition) is a mathematical concept which proved very useful in the analysis of partial differential equations and, more generally, pseudo-differential operators (see, e.g., [21, 14, 22, 23, 35]). The utility of this concept was also realized by quantum field theorists and, in fact, early forms of this notion can be traced in literature on quantum field theory (see e.g. [25, 24] and references cited therein) while the mathematical definition of the  $C^\infty$ -wavefront set in the form as it is used nowadays in mathematics apparently is due to Hörmander [21]. Also, the notion of analytic wavefront set was, parallelly

to its introduction in mathematics [33]<sup>1</sup>, developed in quantum field theory [4, 24] and used there mainly for the study of analytic properties of scattering amplitudes and the behaviour of Wightman distributions in momentum space. However, until recently, microlocal analytic methods were apparently seldom used for the study of Wightman distributions in configuration space. The main reason seems to be that in the Wightman framework of quantum field theory in Minkowski spacetime one has the global Fourier-transform at one's disposal so that the need for a “local” version of Fourier-transform methods, as provided e.g. by the wavefront set-concept, does not automatically arise, at least not when considering theories in vacuum representation.

The situation is, of course, much different when one wishes to study quantum field theory in curved spacetime (see [37, 16] as general references), where one is faced with all the difficulties brought about by the absence of translational symmetries, and in the generic case, the absence of any spacetime symmetry. As is well-known, the notions of a vacuum state, and of a particle, are in quantum field theory in Minkowski spacetime tied to invariance and spectrum condition with respect to the translation group (see e.g. [34, 19, 18, 2]). The spectrum condition is of particular importance since it expresses dynamical stability of quantum field theories. In quantum field theory in Minkowski spacetime, the spectrum condition can be formulated by means of globally Fourier-transforming the action of the translations on the observables and by suitably restricting the resulting Fourier-spectrum of vacuum expectation values. But in curved spacetimes, this is not possible, and one has to look for other ways of formulating conditions of dynamical stability. The wavefront set describes ultra-local remnants of the singular contributions of a distribution in Fourier-space, and this property allows to give appropriate local versions of restrictions on the Fourier-spectrum for dynamically stable states also in quantum field theory in curved spacetime. But even though this line of thought was to some extent developed in the mathematical literature [21, 15], it has only quite recently been investigated seriously within mathematical physics, beginning with Radzikowski's result which says that the two-point function of a state of the free scalar field on a curved (globally hyperbolic) spacetime is of Hadamard form exactly if its wavefront set has a structure which is formally the same as that displayed by the wavefront set of the free field vacuum in Minkowski spacetime [29]. This result bears some importance since several works provide, from various directions, evidence that Hadamard states of the free field in curved spacetime are to be viewed as physical, dynamically stable states (see [37, 16] and references cited there, and also [26, 36]).

Radzikowski then proposed that generally the spectrum condition for Wightman fields in Minkowski spacetime should in curved spacetime be replaced by restrictions on the wavefront sets of the  $n$ -point Wightman functions. His first proposal for such restrictions (called WFSSC, “wavefront set spectrum condition”) underwent some changes [28, 29, 6]; in their current form, they are called  $\mu$ SC, “microlocal spectrum condition” [6]. Subsequently, the use of wavefront set methods and the closely related pseudo-differential operator techniques led to a few interesting results in quantum

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<sup>1</sup>We recommend that the reader consults the Notes to Chapter IX in [21] for a brief but informative review of the development in this branch of mathematics together with a list of the relevant references.

field theory in curved spacetime. Examples are Radzikowski's local-to-global singularity theorem for two-point functions of the free scalar field [30], Junker's proof that adiabatic vacuum states of the Klein-Gordon field are Hadamard states [26], the covariant definition of Wick-products of the scalar field [6] and the program by Brunetti and Fredenhagen which develops the Epstein-Glaser framework for the perturbative construction of interacting quantum field theories and establishes renormalizability of  $\phi_4^4$  on curved spacetime [5]; finally, there are results concerning the stability of quantum fields in spacetimes which contain compactly generated Cauchy-horizons [27] (such spacetimes have been proposed as modelling situations where "time-machines are set into operation" [20]).

These developments suggest to further pursue and utilize wavefront set concepts and techniques in general quantum field theory (both in curved and flat spacetimes). In its present formulation, the notion of the wavefront set applies to distributions and hence to theories which are formulated in terms of pointlike fields and their corresponding Wightman distributions. From the point of view of general local quantum field theory [19, 18] the description of a theory in terms of pointlike fields is, however, not completely intrinsic, and thus a notion of wavefront set which depends on the use of pointlike fields isn't a completely intrinsic concept in general quantum field theory (in operator algebraic formulation) either. We therefore attempt to generalize the notion of wavefront set in such a way that it becomes an intrinsic concept in algebraic quantum field theory, like the spectrum of the translation group of a quantum field theory in Minkowski spacetime. In the present work, we shall in this attempt concentrate on algebraic quantum field theory on Minkowski spacetime. The methods we use here can in principle be generalized to quantum field theories in curved spacetime. In fact, the future application to quantum field theory in curved spacetime is the main motivation for the present study which ought to be seen as a first step in a program which eventually aims at establishing structural results of quantum field theory (e.g. spin-statistics theorems) in curved spacetime with the help of microlocal analytic methods.

The starting point of our work will be the observation (cf. Prop. 2.1) that the wavefront set of a distribution can be characterized in a novel way which emphasizes its role as an asymptotic notion of (Fourier-) spectrum for the action of the translation group. The idea is, roughly, the following. Suppose we have a dual pair  $L', L$  of vector spaces and a representation  $\mathbb{R}^n \ni x \mapsto \tau_x$  of the additive group  $\mathbb{R}^n$  by automorphisms of  $L$ . Assume further that the functions  $x \mapsto u(\tau_x(f))$ ,  $u \in L'$ ,  $f \in L$ , are continuous. The spectrum  $\text{sp}^\tau u$  of an element  $u \in L'$  with respect to the action  $\tau_x$  may then be defined as the support of the Fourier-transform of  $x \mapsto u \circ \tau_x$ , i.e. as the closed union of the supports (in the sense of distributions) of the Fourier-transforms of all functions  $x \mapsto u(\tau_x(f))$ ,  $f \in L$ . This means that  $\xi \in \mathbb{R}^n$  is not contained in the spectrum of  $u$  if one can find some neighbourhood  $V$  of  $\xi$  in  $\mathbb{R}^n$  so that

$$\lim_{\lambda \rightarrow 0} \int e^{-ik \cdot k} h(\lambda x) u(\tau_x(f)) d^n x = 0 \quad (1.1)$$

holds for all  $k \in V$  and all  $f$  in  $L$ , where  $h \in \mathcal{D}(\mathbb{R}^n)$  is a test-function with  $h(0) = 1$ .

It is obvious that in this manner spectral properties of the action of  $\tau_x$  on  $u$  are

tested globally. To illustrate how one may test the Fourier-spectrum behaviour when  $u$  is “asymptotically localized at a point”, we specialize the setting for a moment and take  $L$  to be the space of test-functions  $\mathcal{D}(\mathbb{R}^n)$  and  $L' = \mathcal{D}'(\mathbb{R}^n)$  with the usual action of the translations  $(\tau_x f)(x') = f(x' - x)$ ,  $f \in \mathcal{D}(\mathbb{R}^n)$ . On  $\mathcal{D}(\mathbb{R}^n)$  there act also the dilations  $(\delta_\lambda f)(x') = \lambda^{-n} f(\lambda^{-1} x')$ ,  $\lambda > 0$ . When we act with the induced action  $\delta'_\lambda u = u \circ \delta_\lambda$  on  $u$ , then  $\delta'_\lambda u$  becomes asymptotically concentrated at the origin as  $\lambda$  approaches 0. However, in general the limit of  $\delta'_\lambda u$  for  $\lambda \rightarrow 0$  will not exist. Nevertheless, we may replace  $u$  by  $\delta'_\lambda u$  in (1.1). That means, instead of looking at the limiting behaviour as  $\lambda \rightarrow 0$  of  $\int e^{-ik \cdot x} h(\lambda x) u(\tau_x(f)) d^n x$ , we investigate the asymptotic behaviour of the expression

$$\int e^{-ik \cdot x} h(\lambda x) (\delta'_\lambda u)(\tau_x(f)) d^n x = \lambda^{-n} \int e^{-i\lambda^{-1} k \cdot x} h(x) u(\tau_x(\delta_\lambda f)) d^n x, \quad f \in \mathcal{D}(\mathbb{R}^d), \quad (1.2)$$

as  $\lambda \rightarrow 0$ . It turns out that the point  $(0, \xi)$  is in the complement of the wavefront set of  $u$  exactly if there are a neighbourhood  $V$  of  $\xi$  and an  $h \in \mathcal{D}(\mathbb{R}^n)$  such that for all  $k \in V$  the expression in (1.2) approaches 0 faster than  $O(\lambda^N)$  as  $\lambda \rightarrow 0$  for any  $N \in \mathbb{N}$ , cf. Prop. 2.1. This allows it to interpret the wavefront set of  $u$  as an asymptotic form of the spectrum of  $u$  with respect to translations when  $u$  is asymptotically localized at a point. (This point was here the origin, but that may be changed simply via replacing  $\delta_\lambda$  by  $\tau_{x'} \delta_\lambda$  so that  $u \circ \tau_{x'} \delta_\lambda$  becomes localized at  $x'$  for  $\lambda \rightarrow 0$ .)

Thus it is already apparent that the concept of wavefront set may be generalized from distributions to elements  $u$  in the dual space  $L'$  of a vector space  $L$  on which translations and dilations act (in a suitably continuous manner); it could even be generalized to more general group actions and a microlocal analysis of general automorphism groups could be developed along this line. But we would now like to indicate that the notion of localization is the central one for our concern, the generalization of the wavefront set concept to the setting of algebraic quantum field theory.

The basic structure of a quantum field theory in the operator algebraic setting is that of an inclusion-preserving map  $\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$  assigning to each  $\mathcal{O} \subset \mathbb{R}^d$  of a  $d$ -dimensional spacetime ( $d \geq 2$ ) a  $C^*$ -algebra containing the observables which are localized in the spacetime region  $\mathcal{O}$ , i.e. which can be measured during times and locations within  $\mathcal{O}$  [19, 18]. Such a map is called a *net of local algebras*. In quantum field theory in flat Minkowski spacetime we additionally assume that the translations act by automorphisms on the local algebras. That means there is a representation of the additive group  $\mathbb{R}^d$  by automorphisms  $\alpha_x$ ,  $x \in \mathbb{R}^d$ , on the  $C^*$ -algebra  $\mathcal{A} = \mathcal{A}(\mathbb{R}^d)$  generated by all the local algebras  $\mathcal{A}(\mathcal{O})$  which acts covariantly,

$$\alpha_x(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\mathcal{O} + x), \quad (1.3)$$

and is suitably continuous. By way of comparison we note that the space  $\mathcal{D}(\mathbb{R}^d)$  is also generated by a net of local test-function spaces,  $\mathcal{O} \rightarrow \mathcal{D}(\mathcal{O})$ , and the translations act covariantly:  $\tau_x(\mathcal{D}(\mathcal{O})) = \mathcal{D}(\mathcal{O} + x)$ . On the local test-function spaces we also have the covariant action of the dilations,  $\delta_\lambda(\mathcal{D}(\mathcal{O})) = \mathcal{D}(\lambda \mathcal{O})$ .

However, although there is a special class of quantum field theories admitting also covariant automorphic actions of the dilations on their nets of local algebras, a general quantum field theory describing e.g. massive elementary particles cannot be expected to be in this way dilation-covariant. This appears at first sight to be an obstruction to the attempt of generalizing the asymptotic localization of a distribution at a point to linear functionals and hence, states, on the algebra  $\mathcal{A}$ . But in [10] a method for analyzing the short distance behaviour of states (positive linear functionals) on  $\mathcal{A}$  was developed which circumvents the problem that one lacks a canonical notion of dilations as actions on a generic net of local algebras. (We will not give a systematic account of this “scaling algebra” method in this paper, yet the basic elements of this approach appear in Sec. 4 where our results make contact with material in [10], so the present work is self-contained. The interested reader is referred to the references [10, 11, 7, 8, 9] for further discussion and results of the “scaling algebra” framework.) Roughly, the idea is to collect all functions  $\lambda \rightarrow A_\lambda \in \mathcal{A}$  depending on a positive real scaling parameter  $\lambda$  which are uniformly bounded and have the same localization properties as elements of the local algebras  $\mathcal{A}(\mathcal{O})$  would have under a covariant action of the dilations, that is,  $A_\lambda \in \mathcal{A}(\lambda\mathcal{O})$ ,  $\lambda > 0$ , for some arbitrary bounded region  $\mathcal{O}$ . Such functions will in Sec. 3 be denoted as families  $(A_\lambda)_{\lambda>0}$  and referred to as “testing-families”. The counterpart at the level of test-functions is to consider not only functions of the positive reals into  $\mathcal{D}(\mathbb{R}^d)$  which are of the form  $\lambda \mapsto \delta_\lambda f$ ,  $\lambda > 0$ , for any  $f \in \mathcal{D}(\mathbb{R}^d)$ , but any suitably bounded family  $(f_\lambda)_{\lambda>0}$  with  $f_\lambda \in \mathcal{D}(\lambda\mathcal{O})$  for some bounded region  $\mathcal{O}$ . The main content of Prop. 2.1 is that, when taking in (1.2) all such families  $(f_\lambda)_{\lambda>0}$  in place of  $\delta_\lambda f$ ,  $\lambda > 0$ , one finds that the described criterion for  $(0, \xi)$  to be in the complement of the wavefront set of  $u$  remains valid. This observation motivates our definition in Sec. 3 of the “asymptotic correlation spectrum” of a continuous linear functional  $\varphi$  on  $\mathcal{A}$  as a natural generalization of the concept of the wavefront set of a distribution, and as an asymptotic version of the Fourier-spectrum of  $\varphi$  with respect to the action of the translations in the case where the functional  $\varphi$  is asymptotically localized at (simultaneously) several points in  $\mathbb{R}^d$ .

We should like to point out that the idea of characterizing the wavefront set of a distribution with the help of testing families  $(f_\lambda)_{\lambda>0}$  in a way similar to Prop. 2.1 is not entirely new, it appears e.g. in the description of the “asymptotic frequency set” in [17]. However, our approach is novel in emphasizing the “asymptotic spectrum” point of view, allowing immediate generalization to functionals and group actions on vector spaces. Moreover we remark that the asymptotic correlation spectrum tests spectral properties of the states of a given quantum field theory and not directly those of the corresponding “scaling limit states” and “scaling limit theories” in the sense of [10] although there is a relation, as we discuss in Section 4.

This work is organized as follows. Section 2 establishes the results already mentioned concerning the description of the wavefront set of distributions. In Section 3 we introduce, motivated as indicated by the results of Proposition 2.1, the notion the asymptotic correlation spectrum of a continuous linear functional on  $\mathcal{A}$ . Section 4 is concerned with a study of asymptotic correlation spectra in the setting of quantum field theories in Minkowski spacetime fulfilling locality and spectrum condition. The latter two properties are found to imply “upper bounds” for the asymptotic correla-

tion spectra of physical states. Moreover, constraints on the asymptotic correlation spectra of physical states are shown to imply certain properties of the corresponding “scaling limit states” in the sense of [10]. In Sec. 5 we assume that there is a Wightman field affiliated to a net of local (von Neumann) algebras, and we compare the asymptotic correlation spectrum of a physical state with the wavefront sets of its associated Wightman distributions. It is shown that the wavefront sets of the Wightman functions provide “lower bounds” for the asymptotic correlation spectra. We also show that if the Wightman field is non-trivial, i.e. the field operators are not just multiples of the unit operator, then for each  $n \in \mathbb{N}$  the essential support of the  $2n$ -point distribution associated with any separating state vector in the field domain must contain points of a certain type, and thus has non-empty wavefront set. The article is concluded by summary and outlook in the final Section 6.

## 2 On the wavefront set of distributions

As discussed in the Introduction, we wish to introduce in the present section a characterization of the wavefront set of a distribution which may be viewed as an asymptotic spectrum with respect to the action of the translation group. It bears some reminiscence to the “frequency set” of a distribution introduced by Guillemin and Sternberg [17].

*Notation.* In the following discussion,  $m \in \mathbb{N}$  is arbitrary but kept fixed, thus we write  $\mathcal{D} \equiv \mathcal{D}(\mathbb{R}^m)$ ,  $\mathcal{S} \equiv \mathcal{S}(\mathbb{R}^m)$ ,  $\mathcal{D}' \equiv \mathcal{D}'(\mathbb{R}^m)$ , etc. We denote by  $\tau_y$ ,  $y \in \mathbb{R}^m$ , the action of the translations on test-functions:

$$(\tau_y f)(x) := f(x - y), \quad x, y \in \mathbb{R}^m, \quad f \in \mathcal{D}. \quad (2.1)$$

We often write  $\langle u, f \rangle \equiv u(f)$ ,  $f \in \mathcal{D}$ ,  $u \in \mathcal{D}'$ , for the dual pairing between distributions and test-functions.

The reflection of a test-function  $f$  with respect to the origin will be denoted by

$${}^r f(x) := f(-x), \quad x \in \mathbb{R}^m. \quad (2.2)$$

(In the literature,  $\check{f}$  is often used to denote the reflection of  $f$ .)

The Fourier-transform of a test-function  $f$  is defined by

$$\widehat{f}(k) := \int e^{-ik \cdot x} f(x) d^m x, \quad k \in \mathbb{R}^m, \quad (2.3)$$

where  $k \cdot x$  denotes the Euklidean scalar product of elements in  $\mathbb{R}^m$ .

We also use the following convention which is more or less standard. Let  $\varphi_k : \mathbb{R}^+ \rightarrow \mathbb{C}$  be a family of functions parametrized by elements  $k$  in some set  $K$ . Then the statement that

$$\varphi_k(\lambda) = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad \text{uniformly in } k \in K \quad (2.4)$$

is an abbreviation of the following statement:

For each  $N \in \mathbb{N}$  there exist  $C_N > 0$  and  $\lambda_N > 0$  such that

$$\sup_{k \in K} |\varphi_k(\lambda)| < C_N \cdot \lambda^N \quad \text{for all } 0 < \lambda < \lambda_N. \quad (2.5)$$

It should be observed that if (2.4) holds, then there holds equivalently for all  $\mu \in \mathbb{R}$  and any  $\nu > 0$ ,  $\varphi_k(\lambda) = O^\infty(\lambda^\nu)$  as well as  $\lambda^\mu \varphi_k(\lambda) = O^\infty(\lambda)$  for  $\lambda \rightarrow 0$ , uniformly in  $k \in K$ . It is also not difficult to check e.g. the following: When we have a family of functions  $a_\kappa : \mathbb{R}^+ \rightarrow \mathbb{C}$  indexed by real numbers  $\kappa$  such that  $a_\kappa(\lambda) = O^\infty(\lambda)$  as  $\lambda \rightarrow 0$  for each  $\kappa$ , and if  $b : \mathbb{R}^+ \rightarrow \mathbb{C}$  is another function having the property that there is some  $c \in \mathbb{R}$  and for every  $\nu > 0$  some  $\kappa = \kappa(\nu)$  with  $\lambda^{c\kappa} a_\kappa(\lambda) - b(\lambda) = O(\lambda^\nu)$  as  $\lambda \rightarrow 0$ , then it follows that  $b(\lambda) = O^\infty(\lambda)$  as  $\lambda \rightarrow 0$ .

We shall now introduce a collection of families  $(f_\lambda)_{\lambda>0}$  of test-functions  $f_\lambda$  indexed by a real positive parameter  $\lambda$ ; we will call such families “testing-families”. We shall use the notation  $\mathbf{(f_\lambda)} \equiv (f_\lambda)_{\lambda>0}$ . Observe that thus, by convention, the use of parentheses in bold print means that we are considering the whole testing family (i.e. a mapping from  $\mathbb{R}^+$  into  $\mathcal{D}$ ), as opposed to e.g. writing  $u(f_\lambda)$ , where a distribution  $u$  is evaluated on the member of a testing family at some particular parameter value  $\lambda$ . For  $x \in \mathbb{R}^n$  and  $\mathcal{O}$  a bounded, open neighbourhood of  $0 \in \mathbb{R}^m$ , we define the set

$$\mathbf{F}_x(\mathcal{O}) := \left\{ \mathbf{(f_\lambda)} : f_\lambda \in \mathcal{D}, \text{ supp } f_\lambda \subset \lambda\mathcal{O} + x, \sup_\lambda \|f_\lambda\| < \infty \right\}, \quad (2.6)$$

where  $\|f\| := \sup_{y \in \mathbb{R}^m} |f(y)|$ . As our collection of testing families we then take  $\mathbf{F}_x := \bigcup_{\mathcal{O}} \mathbf{F}_x(\mathcal{O})$ .

We should finally note that we will usually denote variables in configuration space by letters  $x, x', y$  etc., while reserving the letters  $\xi, k, \ell$  for variables in Fourier-space.

Let us also recall the definition of the *wavefront set*  $WF(u)$  of a distribution  $u \in \mathcal{D}$ :  $WF(u)$  is the complement set in  $\mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\})$  of all those pairs  $(x, \xi) \in \mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\})$  having the property that there exists some  $\chi \in \mathcal{D}$  with  $\chi(x) \neq 0$  and an open neighbourhood  $V$  of  $\xi$  (in  $\mathbb{R}^m \setminus \{0\}$ ) such that there holds

$$\sup_{k \in V} |\widehat{\chi u}(\lambda^{-1}k)| = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0. \quad (2.7)$$

Here,  $\widehat{\chi u}$  is the Fourier-transform of the distribution  $\chi u$ , which may be expressed as  $\widehat{\chi u}(k) = u(e_k \chi)$  with  $e_k(y) := e^{-ik \cdot y}$ . This form of definition of the wavefront set can be found in [14]. We refer to this reference and the e.g. the monographs [22, 23, 35] for considerable further discussion on the properties of the wavefront set and its use in studying partial (or pseudo-) differential operators.

**Proposition 2.1.** *Let  $x \in \mathbb{R}^m$ ,  $\xi \in \mathbb{R}^m \setminus \{0\}$ , and  $u \in \mathcal{D}'$ . Then the following statements are equivalent.*

- (a)  $(x, \xi) \notin WF(u)$
- (b) *There exist an open neighbourhood  $V$  of  $\xi$  and an  $h \in \mathcal{D}$  with  $h(0) = 1$ , such that for each family  $\mathbf{(f_\lambda)} \in \mathbf{F}_x$  there holds*

$$\int e^{-i\lambda^{-1}k \cdot y} h(y) \langle u, \tau_y f_\lambda \rangle d^m y = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad (2.8)$$

uniformly in  $k \in V$ .

(c) There exist an open neighbourhood  $V$  of  $\xi$ , an  $h \in \mathcal{D}$  with  $h(0) = 1$ , and some  $g \in \mathcal{D}$  with  $\widehat{g}(0) = 1$  such that for all  $p \geq 1$  it holds that

$$\int e^{-i\lambda^{-1}k \cdot y} h(y) \langle u, \tau_y g_\lambda^{(p)} \rangle d^m y = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad (2.9)$$

uniformly in  $k \in V$ , where

$$g_\lambda^{(p)}(x') := g(\lambda^{-p}(x' - x)), \quad \lambda > 0, x' \in \mathbb{R}^m. \quad (2.10)$$

*Proof.* The first step is to facilitate the proof by demonstrating that it is sufficient to consider the case  $x = 0$ . To this end, we notice that  $(x, \xi) \notin WF(u)$  if and only if  $(0, \xi) \notin WF(u \circ \tau_{-x})$  by the well-known transformation properties of the wavefront set. Secondly, we notice that requiring for  $u$  the condition (2.8) to hold for all  $(f_\lambda) \in \mathbf{F}_x$  is equivalent to demanding that (2.8) holds with  $u \circ \tau_{-x}$  in place of  $u$  for all  $(f_\lambda) \in \mathbf{F}_0$ , as can be seen from

$$\langle u \circ \tau_{-x}, \tau_y f_\lambda \rangle = \langle u, \tau_y \tau_{-x} f_\lambda \rangle \quad (2.11)$$

and the observation that  $(f_\lambda) \mapsto (\tau_{-x} f_\lambda)$  induces a bijective map from  $\mathbf{F}_x$  onto  $\mathbf{F}_0$ . By the same type of argument one concludes that replacing in (2.9)  $u$  by  $u \circ \tau_{-x}$  is equivalent to replacing in the definition (2.9) of  $g_\lambda^{(p)}$  the  $x$  by 0. Thus it suffices to prove the claimed equivalences for the case  $x = 0$ . We will use the notation  $\mathbf{F} \equiv \mathbf{F}_0$ .

To carry on, it is convenient to collect first a few auxiliary results.

**Lemma 2.2.** *( $\alpha$ ) Let  $(f_\lambda) \in \mathbf{F}$ . Then there is  $c > 0$  such that*

$$\sup_{k \in \mathbb{R}^m} |\widehat{f_\lambda}(\lambda^{-1}k)| \leq c \cdot \lambda^m. \quad (2.12)$$

*( $\beta$ ) Let  $w_\lambda$ ,  $1 > \lambda > 0$ , be a family of smooth functions on  $\mathbb{R}^m$  fulfilling the bound*

$$|w_\lambda(\lambda^{-1}k)| \leq c \cdot (|k| + 1 + \lambda^{-1})^q \quad (2.13)$$

*for suitable numbers  $c > 0$  and  $q \in \mathbb{R}$ . Assume additionally that there is an open neighbourhood  $V'$  of some  $\xi \in \mathbb{R}^m \setminus \{0\}$  such that*

$$w_\lambda(\lambda^{-1}k') = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad (2.14)$$

*uniformly in  $k' \in V'$ . Then for each open neighbourhood  $V$  of  $\xi$  contained in any compact subset of  $V'$  and for all  $\phi \in \mathcal{S}$  one has*

$$(\phi * w_\lambda)(\lambda^{-1}k) = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad (2.15)$$

*uniformly in  $k \in V$ .*

*Proof.* ( $\alpha$ ) We have  $(f_\lambda) \in \mathbf{F}(\mathcal{O})$  for some bounded set  $\mathcal{O}$ , hence we obtain

$$|\widehat{f_\lambda}(\lambda^{-1}k)| \leq \left| \int e^{-i\lambda^{-1}k \cdot y} f_\lambda(y) d^m y \right| \leq \text{vol}(\lambda\mathcal{O}) \sup_\lambda \|f_\lambda\|, \quad (2.16)$$

implying the assertion.

( $\beta$ ) Let  $U$  be an open, bounded neighbourhood around the origin in  $\mathbb{R}^m$  and  $V$  a bounded open neighbourhood of  $\xi$  such that  $\overline{V} \subset V'$ . Define two functions  $\chi_U, \chi$  on  $\mathbb{R}^m$  by  $\chi_U =$  characteristic function of  $-U$ ,  $\chi = 1 - \chi_U$ . After a change of variables one gets

$$(\phi * w_\lambda)(\lambda^{-1}k) = \frac{1}{\lambda^{m/2}} \int (\chi_U(\ell) + \chi(\ell)) \phi(\lambda^{-1/2}\ell) w_\lambda(\lambda^{-1}(k - \lambda^{1/2}\ell)) d^m \ell. \quad (2.17)$$

Since  $\phi \in \mathcal{S}$ , it holds that

$$\int |\chi(\ell) \phi(\lambda^{-1/2}\ell)| (\lambda^{-1} + |\ell|)^s d^m \ell = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad (2.18)$$

for all  $s \geq 0$ . Moreover, for sufficiently small  $\lambda$  one has  $V + \lambda^{1/2}U \subset V'$ , thus

$$\sup_{\ell \in \mathbb{R}^m} |\chi_U(\ell) w_\lambda(\lambda^{-1}(k - \lambda^{1/2}\ell))| = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad (2.19)$$

uniformly in  $k \in V$ . This proves the claim.  $\square$

We return to the proof of Proposition 1 and begin with:

(a)  $\Rightarrow$  (b). Let  $(0, \xi) \notin WF(u)$ . Then there exist an open neighbourhood  $V'$  of  $\xi$  and a function  $\chi \in \mathcal{D}$  which is equal to 1 in an open neighbourhood  $U$  of 0, with the property that

$$\widehat{\chi u}(\lambda^{-1}k') = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad (2.20)$$

uniformly in  $k' \in V'$ . Now let  $h \in \mathcal{D}$  with  $h(0) = 1$  and a convex open neighbourhood  $\mathcal{O}$  of 0 be chosen so that  $\text{supp } h + \mathcal{O} \subset U$ . Then we find that the support properties of  $h$  and of members  $(f_\lambda)$  of  $\mathbf{F}(\mathcal{O})$  imply for all  $k \in \mathbb{R}$  and  $1 > \lambda > 0$ ,

$$\begin{aligned} \int e^{-i\lambda^{-1}k \cdot y} h(y) \langle u, \tau_y f_\lambda \rangle d^m y &= \int h(y) e^{-i\lambda^{-1}k \cdot y} \langle \chi u, \tau_y f_\lambda \rangle d^m y \\ &= \widehat{h} * (\widehat{f_\lambda} \cdot \widehat{\chi u})(\lambda^{-1}k). \end{aligned} \quad (2.21)$$

Now observe that  $w_\lambda := \widehat{f_\lambda} \cdot \widehat{\chi u}$  fulfills by Lemma 2( $\alpha$ ) and due to the fact that  $\widehat{\chi u}$  is polynomially bounded the assumptions of Lemma 2( $\beta$ ). Whence we may apply Lemma 2( $\beta$ ) to the effect that the last expression in (2.21) is of order  $O^\infty(\lambda)$  uniformly for  $k$  in some open neighbourhood  $V$  of  $\xi$  as  $\lambda \rightarrow 0$ .

(b)  $\Rightarrow$  (c). Assume that condition (b) holds for suitable choices of  $V$ ,  $h$  and  $\mathcal{O}$ . Let  $g \in \mathcal{D}$  have  $\text{supp } g \subset \mathcal{O}$ . Then for each  $p \geq 1$  the testing-family  $(f_\lambda)$  defined by

$$f_\lambda := g_\lambda^{(p)}, \quad \lambda > 0, \quad (2.22)$$

is contained in  $\mathbf{F}(\mathcal{O})$ , and this implies (c).

(c)  $\Rightarrow$  (a). We now assume that (c) holds with suitable choices of  $V$ ,  $h$  and  $g$ . Then we choose some  $\chi \in \mathcal{D}$  which is equal to 1 on a ball centered around the origin containing the set  $\text{supp } h + \text{supp } g$ . Note also that  $\widehat{g}(0) = 1$  implies  $\widehat{g}(0) = 1$ . Since for some  $c > 0$  and  $q \in \mathbb{R}$  it holds that  $|\widehat{\chi u}(k)| \leq c(|k| + 1)^q$ , we obtain with the help of the mean value theorem and suitable constants  $c_1, c_2 > 0$ ,

$$\begin{aligned}
& \frac{1}{\lambda^{pm}} \int e^{-i\lambda^{-1}k \cdot y} h(y) \langle u, \tau_y g_\lambda^{(p)} \rangle d^m y - \widehat{hu}(\lambda^{-1}k) \\
&= \int \widehat{h}(\ell) (\widehat{g}(\lambda^p(\lambda^{-1}k - \ell)) - 1) \cdot \widehat{\chi u}(\lambda^{-1}k - \ell) d^m \ell \\
&\leq c_1 \int |\widehat{h}(\ell)| \lambda^{p-1} |k - \lambda \ell| \cdot (\lambda^{-1}|k| + |\ell| + 1)^q d^m \ell \\
&\leq c_2 \lambda^{p-1-q} \int |\widehat{h}(\ell)| (1 + |\ell|)^{q+1} d^m \ell \\
&= O(\lambda^{p-(q+1)}) \quad \text{as } \lambda \rightarrow 0
\end{aligned} \tag{2.23}$$

uniformly for  $k$  in any fixed bounded subset of  $\mathbb{R}^m$ . In view of our assumption that (c) holds, this last estimate implies that there is some bounded neighbourhood  $V_1 \subset V$  of  $\xi$  such that for arbitrary  $p \geq 1$ ,

$$\widehat{hu}(\lambda^{-1}k) = O(\lambda^{p-(q+1)}) \quad \text{as } \lambda \rightarrow 0 \tag{2.24}$$

holds uniformly in  $k \in V_1$ . But since  $p \geq 1$  is arbitrary and  $q \in \mathbb{R}$  is fixed, this means that  $(0, \xi) \notin WF(u)$ . This completes the proof.  $\square$

The fact that in (2.8) the Fourier-transform of  $y \mapsto \langle u, \tau_y f_\lambda \rangle$  is “windowed” by the function  $h$  ensures that only local properties of  $u$  near the point  $x$  are tested, and so the role of  $h$  is to localize  $u$  near  $x$ . Thus one expects that the behaviour (2.8) and (2.9) of  $u$  is not changed if  $h$  is replaced by  $\phi \cdot h$  when  $\phi$  is any element in  $C^\infty(\mathbb{R}^m)$ . This turns out to be indeed the case.

**Proposition 2.3.** *Let  $x \in \mathbb{R}^m$ ,  $\xi \in \mathbb{R}^m \setminus \{0\}$  and  $u \in \mathcal{D}'$ . Assume that condition (b) of Proposition 1 is fulfilled with suitable choices of  $V$  and  $h$ . Then condition (b) of Proposition 1 holds also if  $h$  is replaced by  $\phi \cdot h$  for any  $\phi \in C^\infty(\mathbb{R}^d)$ , and if at the same time  $V$  is replaced by any open neighbourhood  $V_1$  of  $\xi$  such that  $\overline{V_1}$  is compact and contained in  $V$ .*

*The analogous statement holds also for condition (c) of Proposition 2.1.*

*Proof.* Again it is sufficient to consider the case  $x = 0$ . Let  $(f_\lambda) \in \mathbf{F}(\mathcal{O})$  and let  $\chi \in \mathcal{D}$  be equal to 1 on a neighbourhood of the closure of  $\mathcal{O} + \text{supp } h$ . Then it is easy to check that the family of smooth functions  $w_\lambda$ ,  $1 > \lambda > 0$ , defined by

$$w_\lambda := \widehat{h} * (\widehat{f_\lambda} \cdot \widehat{\chi u}) \tag{2.25}$$

fulfills the assumptions of Lemma 2( $\beta$ ). Moreover,  $\chi \phi \in \mathcal{D}$ , and we have, using Lemma 2( $\beta$ ), for each open, relatively compact neighbourhood  $V_1$  of  $\xi$  with  $\overline{V_1} \subset V$

that

$$\begin{aligned}
& \int e^{-i\lambda^{-1}k \cdot y} (\phi \cdot h)(y) \langle u, \tau_y f_\lambda \rangle d^m y \\
&= \int e^{-i\lambda^{-1}k \cdot y} (\chi \phi \cdot h)(y) \langle \chi u, \tau_y f_\lambda \rangle d^m y \\
&= \widehat{\chi \phi} * w_\lambda(\lambda^{-1}k') = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0
\end{aligned} \tag{2.26}$$

uniformly in  $k \in V_1$ . The argument for the case of condition (c) of Proposition 1 is analogous.  $\square$

### 3 The asymptotic correlation spectrum

We shall in this section present our definition of the “asymptotic correlation spectrum” which is a generalization of the wavefront set in algebraic quantum field theory. To this end, we must first of all describe the algebraic quantum field theory which we are going to consider, namely, a translation covariant theory on  $d$ -dimensional Minkowski-spacetime where  $d \geq 2$ . More precisely, we assume that we are given a net  $\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$  of  $C^*$ -algebras indexed by the double cone regions  $\mathcal{O} \subset \mathbb{R}^d$ .<sup>2</sup> Thus the map  $\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$  assigns to each double cone  $\mathcal{O}$  a  $C^*$ -algebra such that the condition of isotony holds,

$$\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2). \tag{3.1}$$

It will be recalled that this is motivated by the idea to view  $\mathcal{A}(\mathcal{O})$  as the algebra generated by the observables which can be measured at times and locations in the spacetime region  $\mathcal{O}$ , cf. [19, 18] for further discussion.

Moreover, we make the assumption that there is a representation  $\mathbb{R}^d \ni x \mapsto \alpha_x$  of the translation group acting by automorphisms on  $\mathcal{A} := \overline{\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})}^{C^*}$ , the so-called quasilocal algebra of observables. This representation is required to act covariantly,

$$\alpha_x(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\mathcal{O} + x) \tag{3.2}$$

for all  $x \in \mathbb{R}^d$  and all double cones  $\mathcal{O}$ . A further assumption which we add here is that the group action is strongly continuous, meaning that

$$\|\alpha_x(A) - A\| \rightarrow 0 \quad \text{for } x \rightarrow 0 \tag{3.3}$$

holds for all  $A \in \mathcal{A}$ . (In mathematical terms,  $(\mathcal{A}, \{\alpha_x\}_{x \in \mathbb{R}^d})$  is a  $C^*$ -dynamical system.) As will be pointed out later, this strong continuity requirement is not really necessary and could in more special situations be replaced by weaker versions. However, when starting with the  $C^*$ -algebraic setting, it is a natural assumption.

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<sup>2</sup>A double cone  $\mathcal{O}$  is a set of the form  $\mathcal{O} = (V_+ + x) \cap (-V_+ + y)$  and  $y \in V_+$  where  $V_+ = \{(x^0, x^1, \dots, x^{d-1}) \in \mathbb{R}^d : (x^0)^2 - \sum_{j=1}^{d-1} (x^j)^2 > 0, x^0 > 0\}$  is the open forward lightcone;  $\overline{V}_+$  is its closure.

Other assumptions which are standard in quantum field theory like locality and existence of a vacuum state (see [19, 18]) are not needed for the moment but will be introduced later.

With the given theory  $(\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}), \{\alpha_x\}_{x \in \mathbb{R}^d})$  we can now associate testing-families  $(A_\lambda)_{\lambda > 0}$  in the following way: We define for each double cone region  $\mathcal{O}$  in  $\mathbb{R}^d$  and  $x \in \mathbb{R}^d$  the set

$$\mathbf{A}_x(\mathcal{O}) := \left\{ (A_\lambda)_{\lambda > 0} : A_\lambda \in \mathcal{A}(\lambda\mathcal{O} + x), \sup_{\lambda > 0} \|A_\lambda\| < \infty, A_\lambda = 0 \text{ for large } \lambda \right\}. \quad (3.4)$$

The precise meaning of  $A_\lambda = 0$  for large  $\lambda$  in (3.4) is: For each  $(A_\lambda)_{\lambda > 0} \in \mathbf{A}_x(\mathcal{O})$  there is some positive number  $\lambda_A$  such that  $A_\lambda = 0$  if  $\lambda > \lambda_A$ . (This requirement is not central but turns out to be convenient.) As in the case of testing-families of test-functions, we use the notation  $\mathbf{(A}_\lambda)$  to denote the testing family  $(A_\lambda)_{\lambda > 0}$ . We note that  $\mathbf{A}(\mathcal{O})$  is a  $C^*$ -algebra upon defining the algebraic operations pointwise for each scaling-parameter  $\lambda$  (i.e.,  $\mathbf{a(A}_\lambda) + \mathbf{(B}_\lambda) = \mathbf{(aA}_\lambda + B_\lambda)$ ,  $\mathbf{(A}_\lambda) \cdot \mathbf{(B}_\lambda) = \mathbf{(A}_\lambda \cdot B_\lambda)$ ,  $\mathbf{(A}_\lambda)^* = \mathbf{(A}_\lambda^*)$ ) and taking as  $C^*$ -norm  $\|\mathbf{(A}_\lambda)\| = \sup_{\lambda > 0} \|A_\lambda\|$ . It is also clear that the map  $\mathcal{O} \rightarrow \mathbf{A}_x(\mathcal{O})$  is a net of  $C^*$ -algebras since the condition of isotony holds, i.e.  $\mathbf{A}_x(\mathcal{O}_1) \subset \mathbf{A}_x(\mathcal{O}_2)$  for  $\mathcal{O}_1 \subset \mathcal{O}_2$ . Thus our testing-families will be the elements in the  $*$ -algebra  $\mathbf{A}_x := \bigcup_{\mathcal{O}} \mathbf{A}_x(\mathcal{O})$ . It should be noted that we do not take a closure of this set. Observe that

$$\alpha_y(\mathbf{A}_x(\mathcal{O})) = \mathbf{A}_{x+y}(\mathcal{O}) \quad (3.5)$$

holds for all  $x, y \in \mathbb{R}^d$  and double cones  $\mathcal{O}$ , where

$$\alpha_y(A_\lambda) := \mathbf{(A}_y(A_\lambda)). \quad (3.6)$$

In the same way as we have used the testing-families in  $\mathbf{F}_x$  to probe the frequency behaviour of a distribution (a continuous linear functional on the test-function space) infinitesimally close to the point  $x$  in coordinate space, we shall now employ the elements of the scaling algebra  $\mathbf{A}_x$  to analyse the frequency behaviour of a continuous linear functional on  $\mathcal{A}$  close to  $x$ . To do so, we need to introduce further notation. We will generically abbreviate an element  $(x_1, \dots, x_n; k_1, \dots, k_n) \in \mathbb{R}^{dn} \times (\mathbb{R}^{dn} \setminus \{0\})$  by  $(\mathbf{x}; \mathbf{k})$ . With this convention,  $\mathbf{k} \cdot \mathbf{y}$  denotes the scalar product  $\sum_{j=1}^n k_j \cdot y_j$  where in the sum appear the scalar products of the vectors  $k_j$  and  $y_j$  in  $\mathbb{R}^d$ ; it should be borne in mind that the lower indices here aren't coordinate indices. The integration measure  $d^d y_1 \cdots d^d y_n$  will be abbreviated by  $d\mathbf{y}$ . When  $\mathbf{(A}_\lambda^{(1)}) \otimes \cdots \otimes \mathbf{(A}_\lambda^{(n)})$  is a simple tensor in  $\mathbf{A}_{x_1} \otimes \cdots \otimes \mathbf{A}_{x_n}$ , then we denote this relation simply by  $\mathbf{(A}_\lambda^{(\mathbf{x})}) \in \mathbf{A}_\mathbf{x}$ , understanding that  $\mathbf{x} = (x_1, \dots, x_n)$ .

**Definition 3.1.** Let  $\varphi$  be a continuous linear functional on  $\mathcal{A}$ , and let  $n \in \mathbb{N}$ . Then  $ACS^n(\varphi)$ , the  $n$ -th order asymptotic correlation spectrum of  $\varphi$ , is defined as the complement in  $\mathbb{R}^{dn} \times (\mathbb{R}^{dn} \setminus \{0\})$  of all those  $(\mathbf{x}; \boldsymbol{\xi})$  which have the following property: There is an  $h \in \mathcal{D}(\mathbb{R}^{dn})$  with  $h(0) = 1$  and an open neighbourhood  $V$  of  $\boldsymbol{\xi}$  such that

for each  $(A_\lambda^{(\mathbf{x})}) \in \mathbf{A}_\mathbf{x}$  it holds that

$$\int e^{-i\lambda^{-1}\mathbf{k} \cdot \mathbf{y}} h(\mathbf{y}) \varphi(\alpha_{y_1}(A_\lambda^{(1)}) \cdots \alpha_{y_n}(A_\lambda^{(n)})) d\mathbf{y} = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad (3.7)$$

uniformly in  $\mathbf{k} \in V$ .

*Remark.* Since  $\mathcal{A}$  is an algebra,  $\varphi$  can be viewed as a linear map  $\varphi_\otimes : \bigotimes^n \mathcal{A} \rightarrow \mathbb{C}$ ,  $\varphi_\otimes(A_1 \otimes \cdots \otimes A_n) = \varphi(A_1 \cdots A_n)$ . It is thus clear that the definition of  $ACS^n(\varphi)$  for linear functionals  $\varphi$  on  $\mathcal{A}$  generalizes, in an obvious manner, to  $ACS^n(v)$  for any linear functional  $v : \bigotimes^n \mathcal{A} \rightarrow \mathbb{C}$  continuous on the simple tensors.

We shall next collect a few immediate properties of  $ACS^n(\varphi)$  which are reminiscent of corresponding properties of the wavefront set.

**Proposition 3.2.**

(a) (*Analogue of Prop. 2.3.*) If (3.7) holds for some choice of  $h$  and  $V$ , then it holds also with  $\phi \cdot h$  in place of  $h$  for all  $\phi \in C^\infty(\mathbb{R}^{dn})$  and with  $V$  replaced by any open neighbourhood  $V_1$  of  $\boldsymbol{\xi}$  fulfilling  $\overline{V_1} \subset V$ .

(b)  $ACS^n(\varphi)$  is a closed subset of  $\mathbb{R}^{dn} \times (\mathbb{R}^{dn} \setminus \{0\})$  which is conic in the Fourier-space variables (this means that  $(\mathbf{x}; \boldsymbol{\xi}) \in ACS^n(\varphi)$  iff  $(\mathbf{x}; \mu\boldsymbol{\xi}) \in ACS^n(\varphi)$  for all  $\mu > 0$ .)

(c) Translation covariance:  $(\mathbf{x}; \boldsymbol{\xi}) \in ACS^n(\varphi)$  if and only if  $(0; \boldsymbol{\xi}) \in ACS^n(\varphi_\mathbf{x})$  where for each  $\mathbf{x} \in \mathbb{R}^{dn}$ , we define  $\varphi_\mathbf{x} : \bigotimes^n \mathcal{A} \rightarrow \mathbb{C}$  by  $\varphi_\mathbf{x}(A_1 \otimes \cdots \otimes A_n) = \varphi(\alpha_{x_1}(A_1) \cdots \alpha_{x_n}(A_n))$  (cf. the Remark above).

(d) Suppose that the functional  $\varphi$  is Hermitean, i.e. there holds  $\varphi(A^*) = \overline{\varphi(A)}$ ,  $A \in \mathcal{A}$ . Then we have

$$(\mathbf{x}; \boldsymbol{\xi}) \in ACS^n(\varphi) \Leftrightarrow (\bar{\mathbf{x}}; -\boldsymbol{\xi}) \in ACS^n(\varphi) \quad (3.8)$$

with  $\bar{\mathbf{x}} := (x_n, \dots, x_1)$  for each  $\mathbf{x} = (x_1, \dots, x_n)$ .

(e)  $ACS^n(\varphi + \varphi') \subset ACS^n(\varphi) \cup ACS^n(\varphi')$  holds for all continuous linear functionals  $\varphi, \varphi'$  on  $\mathcal{A}$ .

*Proof.*

(a) Given  $(A_\lambda^{(\mathbf{x})}) \in \mathbf{A}_\mathbf{x}$ , we abbreviate:

$$\varphi_\lambda(\mathbf{y}) := \varphi(\alpha_{y_1}(A_\lambda^{(1)}) \cdots \alpha_{y_n}(A_\lambda^{(n)})). \quad (3.9)$$

Then we observe that that  $\varphi_\lambda, \lambda > 0$ , is a uniformly bounded family of continuous functions, and thus  $w_\lambda(\mathbf{k}) := \widehat{h\varphi_\lambda}(\mathbf{k})$  is a family of smooth functions satisfying the assumptions of Lemma 2.2( $\beta$ ). The statement follows then by Lemma 2.2( $\beta$ ) upon noticing that  $\phi$  may be replaced by  $\chi\phi$  for any  $\chi \in \mathcal{D}(\mathbb{R}^{dn})$  with  $\chi = 1$  on  $\text{supp } h$ .

(b) To show that  $ACS^n(\varphi)$  is closed amounts to showing that, if  $(\mathbf{x}; \boldsymbol{\xi}) \notin ACS^n(\varphi)$ , then there are open neighbourhoods  $U$  of  $\mathbf{x}$  and  $W$  of  $\boldsymbol{\xi}$  so that  $(\mathbf{x}'; \boldsymbol{\xi}') \notin ACS^n(\varphi)$  for all  $\mathbf{x}' \in U$  and  $\boldsymbol{\xi}' \in W$ .

So let  $(\mathbf{x}; \boldsymbol{\xi}) \notin ACS^n(\varphi)$ . This means that (3.7) holds with suitable choices of  $V$  and  $h$ , where the function  $h$  is greater than some strictly positive constant in some neighbourhood  $N$  of  $\mathbf{y} = 0$ . Let  $N_1$  and  $N_2$  be two other neighbourhoods of  $0 \in \mathbb{R}^{dn}$  such that  $N_1 + N_2 \subset N$ . Then choose some  $h_1 \in \mathcal{D}(\mathbb{R}^{dn})$  with  $\text{supp } h_1 \subset N_1$  and such that  $h_1(0) = 1$ . Now for all  $(A_\lambda^{\mathbf{x}}) \in \mathbf{A}_{\mathbf{x}}$  and all  $\mathbf{y}' \in N_2$ , one obtains

$$\begin{aligned} & \int e^{-i\lambda^{-1}\mathbf{k} \cdot \mathbf{y}} h_1(\mathbf{y}) \varphi(\alpha_{y_1-y'_1}(A_\lambda^{(1)}) \cdots \alpha_{y_n-y'_n}(A_\lambda^{(n)})) d\mathbf{y} \\ &= e^{-i\lambda^{-1}\mathbf{k} \cdot \mathbf{y}'} \int e^{-i\lambda^{-1}\mathbf{k} \cdot \mathbf{y}} h_1(\mathbf{y} + \mathbf{y}') \varphi(\alpha_{y_1}(A_\lambda^{(1)}) \cdots \alpha_{y_n}(A_\lambda^{(n)})) d\mathbf{y} = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \end{aligned} \quad (3.10)$$

uniformly in  $\mathbf{k} \in V_1$  for some open neighbourhood  $V_1$  of  $\boldsymbol{\xi}$ , since by construction,  $h_1$  has the property that there is for each  $\mathbf{y}' \in N_1$  a  $\phi_{\mathbf{y}'} \in C^\infty(\mathbb{R}^{dn})$  with  $(\phi_{\mathbf{y}'} h)(\cdot) = h_1(\cdot)$ . By the covariance property (3.5), this implies that we have for all  $\mathbf{x}'$  in the open neighbourhood  $U = \{\mathbf{x} - \mathbf{y}' : \mathbf{y}' \in N_2\}$  of  $\mathbf{x}$

$$\int e^{-i\lambda^{-1}\mathbf{k} \cdot \mathbf{y}} h_1(\mathbf{y}) \varphi(\alpha_{y_1}(A_\lambda^{(1)}) \cdots \alpha_{y_n}(A_\lambda^{(n)})) = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad (3.11)$$

uniformly in  $\mathbf{k} \in V_1$  whenever  $(A_\lambda^{(\mathbf{x}')} ) \in \mathbf{A}_{\mathbf{x}'}$ . Since  $V_1$  is an open neighbourhood of  $\boldsymbol{\xi}$ , it is clear that we can also find an open neighbourhood  $W$  of  $\boldsymbol{\xi}$  such that each  $\boldsymbol{\xi}' \in W$  possesses some open neighbourhood  $V_{\boldsymbol{\xi}'} \subset V_1$ . Hence, for each  $\mathbf{x}' \in U$  and  $\boldsymbol{\xi}' \in W$  condition (3.11) holds for all  $(A_\lambda^{(\mathbf{x}')} ) \in \mathbf{A}_{\mathbf{x}'}$  uniformly in  $\mathbf{k} \in V_{\boldsymbol{\xi}'}$ . This shows  $(\mathbf{x}'; \boldsymbol{\xi}') \notin ACS^n(\varphi)$  for  $\mathbf{x}' \in U$ ,  $\boldsymbol{\xi}' \in W$ .

Next we show that  $(\mathbf{x}; \boldsymbol{\xi}) \notin ACS^n(\varphi)$  implies  $(\mathbf{x}; \mu\boldsymbol{\xi}) \notin ACS^n(\varphi)$ , thus establishing the conicity of  $ACS^n(\varphi)$  in the Fourier-space variables. If  $(\mathbf{x}; \boldsymbol{\xi}) \notin ACS^n(\varphi)$ , then we have for all  $(A_\lambda^{(\mathbf{x})}) \in \mathbf{A}_{\mathbf{x}}$ , using the notation (3.9),

$$\int e^{-i\lambda^{-1}\mathbf{k} \cdot \mathbf{y}} h(\mathbf{y}) \varphi_\lambda(\mathbf{y}) d\mathbf{y} = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad (3.12)$$

uniformly in  $\mathbf{k} \in V$  for suitable  $h$ . Setting  $\rho = \mu^{-1} > 0$ , we replace on the left hand side of the last equation the parameter  $\lambda$  by  $\rho\lambda$ . Denoting  $(A_{\rho\lambda}^{(\mathbf{x})})$  by  $(A'_\lambda{}^{(\mathbf{x})})$ , this yields

$$\int e^{-i\lambda^{-1}\mu\mathbf{k} \cdot \mathbf{y}} h(\mathbf{y}) \varphi(\alpha_{y_1}(A'_\lambda{}^{(1)}) \cdots \alpha_{y_n}(A'_\lambda{}^{(n)})) d\mathbf{y} = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad (3.13)$$

uniformly in  $\mathbf{k} \in V$  for all  $(A'_\lambda{}^{(\mathbf{x})}) \in \mathbf{A}_{\mathbf{x}}$ , since  $\mathbf{A}_{\mathbf{x}}$  is invariant under the scale-transformations  $(A_\lambda) \mapsto (A_{\rho\lambda})$ ,  $\rho > 0$ . Hence  $(\mathbf{x}; \mu\boldsymbol{\xi}) \notin ACS^n(\varphi)$ .

(c) This is simply a consequence of (3.5).

(d) The claimed property is easily verified by inspection. (e) is obvious.  $\square$

## 4 The ACS in algebraic quantum field theory

In Section 3, we have described a “theory” just by an inclusion-preserving map  $\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$  assigning  $C^*$ -algebras to double cone regions together with a covariant,

strongly continuous action  $\{\alpha_x\}_{x \in \mathbb{R}^d}$  of the translation group by automorphisms. Now we shall add more structure which is characteristic of quantum field theory proper — like locality and the spectrum condition — and investigate what properties of  $ACSn(\varphi)$  for functionals or states  $\varphi$  on  $\mathcal{A} = \overline{\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})}^{C^*}$  ensue. We will also deduce some consequences resulting from imposing certain “upper bounds” on the shape of  $ACSn(\varphi)$ .

The first relevant assumption we add to a theory  $(\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}), \{\alpha_x\}_{x \in \mathbb{R}^d})$  with the properties listed at the beginning of Section 3.1 is:

(SC) The theory  $(\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}), \{\alpha_x\}_{x \in \mathbb{R}^d})$  is given in a covariant representation satisfying the *spectrum condition*. That means,  $\mathcal{A}$  is an algebra of bounded operators on a Hilbertspace  $\mathcal{H}$ , and there is a weakly continuous representation  $\mathbb{R}^d \ni x \mapsto U(x)$  of the translation group by unitary operators on  $\mathcal{H}$  such that  $\alpha_x(A) = U(x)AU(x)^{-1}$ ,  $x \in \mathbb{R}^d$ ,  $A \in \mathcal{A}$ . Moreover, it holds that the spectrum of  $P = (P_\mu)_{\mu=0, \dots, d-1}$ , the generator of  $U(x) = e^{iP_\mu x^\mu}$ ,  $x = (x^\mu)_{\mu=0, \dots, d-1}$ , is contained in the  $d$ -dimensional closed forward lightcone  $\overline{V}_+$ . (The existence of a vacuum vector is not assumed here.) Note that in the presence of (SC) the unitary group  $U(x)$ ,  $x \in \mathbb{R}^d$ , may be chosen to be contained in  $\mathcal{A}''$  [3, 1] (see also [2, Chp. II] and [32, Prop. 2.4.4]) and we shall henceforth assume that such a choice has been made.

The next point is to define a class of states, or functionals, on  $\mathcal{A}$  which we wish to investigate. In the presence of (SC), these are the continuous functionals on  $\mathcal{A}$  which are normal, i.e. they admit a normal extension to  $\mathcal{A}''$ . Furthermore, we demand that the functionals are “ $C^\infty$  for the energy”. One can define several versions of this property. To present ours, we use the standard notation  $D_x^\beta = i \frac{\partial^{\beta_0}}{\partial x^0} \cdots i \frac{\partial^{\beta_{d-1}}}{\partial x^{d-1}}$  for iterated partial derivatives, where  $\beta = (\beta_0, \dots, \beta_{d-1}) \in \mathbb{N}_0^d$  is a multi-index.

( $s - C^\infty$ ) A continuous, normal functional  $\varphi$  on  $\mathcal{A}$  is called *strongly  $C^\infty$*  ( $s - C^\infty$ ) if the partial derivatives

$$D_x^\beta D_y^\gamma \varphi(U(x)AU(-y)), \quad A \in \mathcal{A}, \quad x, y \in \mathbb{R}^d, \quad (4.1)$$

exist for all multi-indices  $\beta, \gamma$  and induce normal functionals on  $\mathcal{A}$ .

*Remarks.* (i) Standard examples of strongly  $C^\infty$  functionals may be obtained from  $C^\infty$  vectors for the energy, i.e. such vectors  $\psi \in \mathcal{H}$  which are contained in the domain of  $(P_0)^N$  for all  $N \in \mathbb{N}$ . In view of the spectrum condition,  $\psi$  lies then also in the domain of any power of  $P_\mu$ ,  $\mu = 1, \dots, d-1$ . Any two  $C^\infty$  vectors  $\psi', \psi$  give rise to a normal, strongly  $C^\infty$  functional  $\varphi(A) = \langle \psi', A\psi \rangle$ ,  $A \in \mathcal{A}$ .

(ii) An equivalent way of expressing the  $s - C^\infty$  property — which we will make use of — is the following, as may easily be checked: When we denote by  $F_{\varphi, A}$  the Fourier transform of the function  $(x_1, x_2) \mapsto \varphi(U(x_1)AU(-x_2))$  (this function is a tempered distribution, and so is its Fourier transform), it follows that for any  $N \in \mathbb{N}$  and any  $\phi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$  there is some constant  $c > 0$  so that

$$|\phi * F_{\varphi, A}(k_1, k_2)| (1 + |k_1| + |k_2|)^N \leq c \cdot \|A\|, \quad A \in \mathcal{A}, \quad k_1, k_2 \in \mathbb{R}^d. \quad (4.2)$$

The formulation of the subsequent result is preceded by a list of notational conven-

tions: For a continuous linear functional  $\varphi$  on  $\mathcal{A}$  and we write <sup>3</sup>

$$ACS_{\mathbf{x}}^n(\varphi) := \{\mathbf{k} \in \mathbb{R}^{dn} \setminus \{0\} : (\mathbf{x}; \mathbf{k}) \in ACS^n(\varphi)\}, \quad \mathbf{x} \in \mathbb{R}^{dn}, \quad (4.3)$$

$$\pi_2 ACS^n(\varphi) := \left[ \bigcup_{\mathbf{x} \in \mathbb{R}^{dn}} ACS_{\mathbf{x}}^n(\varphi) \right]^-. \quad (4.4)$$

We denote  $n$ -tuples of vectors in  $\mathbb{R}^d$  again by  $\mathbf{k} = (k_1, \dots, k_n)$ , and set

$$k^{[j]} := \sum_{i=j}^n k_i, \quad j = 1, \dots, n. \quad (4.5)$$

Then we define the set

$$\mathcal{V}_n := \{(k_1, \dots, k_n) \in \mathbb{R}^{dn} : k^{[j]} \in \overline{V}_+, j = 2, \dots, n, k^{[1]} = 0\}. \quad (4.6)$$

Thus  $\mathcal{V}_n$  coincides with the bound for the support of the Fourier-transformed  $n$ -point vacuum expectation values in quantum field theory [34].

**Proposition 4.1.** *Suppose that the theory satisfies the spectrum condition (SC), and let  $\varphi$  be a continuous, normal functional on  $\mathcal{A}$ . If  $\varphi$  is strongly  $C^\infty$  then*

$$\pi_2 ACS^n(\varphi) \subset \mathcal{V}_n \setminus \{0\} \quad \text{for all } n \in \mathbb{N}.$$

*Proof.*  $\mathcal{V}_n$  is a closed set in  $\mathbb{R}^{dn}$ . We set  $\mathcal{X}_n := \mathbb{R}^{dn} \setminus \mathcal{V}_n$  which is open both in  $\mathbb{R}^{dn}$  and  $\mathbb{R}^{dn} \setminus \{0\}$ . To prove the Proposition, it suffices to show that given  $\mathbf{x} \in \mathbb{R}^{dn}$  and  $\boldsymbol{\xi} \in \mathcal{X}_n$ , there is an open neighbourhood  $V$  of  $\boldsymbol{\xi}$  and some  $h \in \mathcal{D}(\mathbb{R}^{dn})$  with  $h(0) = 1$  so that for all  $(A_\lambda^{(\mathbf{x})}) \in \mathbf{A}_{\mathbf{x}}$  there holds

$$\int e^{-i\lambda^{-1}\mathbf{k} \cdot \mathbf{y}} h(\mathbf{y}) \varphi_\lambda(\mathbf{y}) d\mathbf{y} = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow \infty \quad (4.7)$$

uniformly in  $\mathbf{k} \in V$ ; our by now familiar abbreviation

$$\varphi_\lambda(\mathbf{y}) = \varphi(\alpha_{y_1}(A_\lambda^{(1)}) \cdots \alpha_{y_n}(A_\lambda^{(n)})) \quad (4.8)$$

will be recalled.

In the following, there will often appear  $n+1$ -tuples of vectors in  $\mathbb{R}^d$  which will be denoted

$$\underline{z} = (\mathbf{z}, z_{n+1}) = (z_1, \dots, z_{n+1}). \quad (4.9)$$

To exploit the spectrum condition, it is customary to pass from the variable  $\mathbf{y}$  in (4.8) to relative variables  $z_1 = y_1$ ,  $z_2 = y_2 - y_1$ ,  $z_n = y_n - y_{n-1}$ . In this way one obtains, upon setting

$$\Psi_\lambda(\underline{z}) := \varphi(U(z_1)A_\lambda^{(1)}U(z_2)A_\lambda^{(2)}U(z_3) \cdots U(z_n)A_\lambda^{(n)}U(-z_{n+1})), \quad (4.10)$$

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<sup>3</sup>The closure in (4.4) is understood in the set  $\mathbb{R}^d \setminus \{0\}$ .

that

$$\Phi_\lambda(\mathbf{z}) := \Psi_\lambda(\mathbf{z}, -\sum_{j=1}^n z_j) = \varphi_\lambda(\mathbf{y}), \quad (4.11)$$

and similarly, with  $g(\mathbf{z}) = h(z_1, z_1 + z_2, \dots, \sum_{j=1}^n z_j)$ ,

$$\int e^{-i\lambda^{-1} \sum_{j=1}^n k^{[j]} \cdot z_j} g(\mathbf{z}) \Phi_\lambda(\mathbf{z}) d\mathbf{z} = \int e^{-i\lambda^{-1} \mathbf{k} \cdot \mathbf{y}} h(\mathbf{y}) \varphi_\lambda(\mathbf{y}) d\mathbf{y}. \quad (4.12)$$

Now let  $\Gamma_n := \{0\} \times \overline{V}_+ \times \dots \times \overline{V}_+ \subset (\mathbb{R}^d)^n$  (the set  $\overline{V}_+$  appears  $n-1$  times) and  $R_n := (\mathbb{R}^d)^n \setminus \Gamma_n$ . Observe that we have

$$(k^{[1]}, \dots, k^{[n]}) \in \Gamma_n \Leftrightarrow (k_1, \dots, k_n) \in \mathcal{V}_n. \quad (4.13)$$

We will now demonstrate that given any conic subset  $E \subset R_n$  which is closed in  $\mathbb{R}^{dn} \setminus \{0\}$ , any  $g \in \mathcal{D}(\mathbb{R}^{dn})$  and any  $(\mathbf{A}_\lambda^{(\mathbf{x})}) \in \mathbf{A}_\mathbf{x}$  for arbitrary  $\mathbf{x} \in \mathbb{R}^{dn}$ , one can find for each  $N \in \mathbb{N}$  some number  $c > 0$  such that

$$\sup_\lambda \sup_{\mathbf{k} \in E} |\widehat{g\Phi_\lambda}(\mathbf{k})| (1 + |\mathbf{k}|)^N \leq c. \quad (4.14)$$

This property can be seen to imply, in view of (4.5,12,13), the required relation (4.7). To prove (4.14) one first observes that the assumptions entail the following properties of  $\widehat{\Psi}_\lambda$ , the Fourier-transform of  $\Psi_\lambda$ : Roughly speaking,  $\widehat{\Psi}_\lambda(k_1, \dots, k_{n+1})$  is rapidly decreasing in the first and last entries  $k_1$  and  $k_{n+1}$  (implied by the  $s - C^\infty$  property), and has support in  $\overline{V}_+$  with respect to each of the remaining variables  $k_2, \dots, k_n$  (implied by (SC)). Moreover, these properties are uniform in  $\lambda$ . But we must take into account that  $\widehat{\Psi}_\lambda$  is actually a distribution, requiring a slightly different formulation of these properties. So let  $\Gamma'_{n+1} := \{0\} \times \overline{V}_+ \times \dots \times \overline{V}_+ \times \{0\} = \Gamma_n \times \{0\} \subset (\mathbb{R}^d)^{n+1}$  (the set  $\overline{V}_+$  appears again  $n-1$  times) and  $R'_{n+1} := (\mathbb{R}^d)^{n+1} \setminus \Gamma'_{n+1}$ . What we will show is that given any conic subset  $E' \subset R'_{n+1}$  which is closed in  $\mathbb{R}^{d(n+1)} \setminus \{0\}$ , any  $\chi \in \mathcal{D}(\mathbb{R}^{d(n+1)})$  and any  $(\mathbf{A}_\lambda^{(\mathbf{x})}) \in \mathbf{A}_\mathbf{x}$  one can find for every  $N \in \mathbb{N}$  some constant  $c' > 0$  with

$$\sup_\lambda \sup_{\underline{\mathbf{k}} \in E'} |\widehat{\chi\Psi_\lambda}(\underline{\mathbf{k}})| (1 + |\underline{\mathbf{k}}|)^N \leq c'. \quad (4.15)$$

Let us point out how this property entails (4.14). Define  $Q : \mathbb{R}^{dn} \rightarrow \mathbb{R}^{d(n+1)}$  by  $Q(\mathbf{z}) := (\mathbf{z}, -\sum_{j=1}^n z_j)$ . The derivative  $DQ$  of this map is constant, and its transpose is given by  ${}^t(DQ)\underline{\mathbf{k}} = (k_1 - k_{n+1}, \dots, k_n - k_{n+1})$ . The set

$$N_Q = \{(Q(\mathbf{z}), \underline{\mathbf{k}}) \in \mathbb{R}^{d(n+1)} \times \mathbb{R}^{d(n+1)} : {}^t(DQ)\underline{\mathbf{k}} = 0\}$$

is therefore contained in  $\mathbb{R}^{d(n+1)} \times \Delta_{n+1}$  where  $\Delta_{n+1}$  is the total diagonal in  $(\mathbb{R}^d)^{n+1}$ . Since  $\Delta_{n+1} \cap (\Gamma'_{n+1} \setminus \{0\}) = \emptyset$ , we see that

$$N_Q \cap [\mathbb{R}^{d(n+1)} \times (\Gamma'_{n+1} \setminus \{0\})] = \emptyset.$$

Observe also that  ${}^t(DQ)(\Gamma'_{n+1} \setminus \{0\}) = \Gamma_n \setminus \{0\}$ . Thus we can apply Theorem 8.2.4 in [22] which says, for our situation, that (4.15) implies for any conic subset  $E$  of  $\mathbb{R}^{dn} \setminus \{0\}$  with  $\overline{E} \subset R_n$  the relation

$$\sup_{\lambda} \sup_{\mathbf{k} \in E} |(\widehat{(\chi \Psi_{\lambda} \circ Q)})(\mathbf{k})| (1 + |\mathbf{k}|)^N < c_N \quad (4.16)$$

for all  $N \in \mathbb{N}$  with suitable constants  $c_N > 0$ . Since  $\Psi_{\lambda} \circ Q = \Phi_{\lambda}$ , one deduces (4.14) from (4.16).

So we are left with having to prove relation (4.15). The proof proceeds by a variation of more or less standard arguments which can be found in slightly different forms in the literature, e.g. [22, Chp. VIII]. To begin with, we have that  $\sup_{\lambda} \|\Psi_{\lambda}\|_{\infty} < a$  for some  $a > 0$ , thus  $\widehat{\psi} \Psi_{\lambda} \in L^1$  and  $\sup_{\lambda} \|\psi * \widehat{\Psi}_{\lambda}\|_{\infty} \leq b$  for some  $b > 0$  whenever  $\psi \in \mathcal{S}(\mathbb{R}^{d(n+1)})$ . Consequently, if  $\rho_{\mu}(\underline{\mathbf{k}}) = \rho(\underline{\mathbf{k}}/\mu)$ ,  $\underline{\mathbf{k}} \in \mathbb{R}^{d(n+1)}$ ,  $\mu > 0$ , where  $\rho \in \mathcal{D}(\mathbb{R}^{d(n+1)})$ ,  $0 \leq \rho \leq 1$  and  $\rho$  is equal to 1 on an arbitrary open ball containing the origin, we obtain for any  $\phi \in \mathcal{S}(\mathbb{R}^{d(n+1)})$ , any  $N \in \mathbb{N}$  and any  $\eta > 1$

$$\sup_{\lambda} \sup_{\underline{\mathbf{k}} \in \mathbb{R}^{d(n+1)}} |\psi * \widehat{\Psi}_{\lambda}(\tau_{\eta \underline{\mathbf{k}}}(\phi - \rho_{\underline{\mathbf{k}}} \phi))| (1 + |\underline{\mathbf{k}}|)^N \leq C_N \quad (4.17)$$

for suitable  $C_N > 0$ . To see this, let  $s > 0$  be the radius of the open ball around the origin where  $\rho = 1$ . Then consider

$$\begin{aligned} & |\psi * \widehat{\Psi}_{\lambda}(\tau_{\eta \underline{\mathbf{k}}}(\phi - \rho_{\underline{\mathbf{k}}} \phi))| (1 + |\underline{\mathbf{k}}|)^N \\ & \leq b \int |\phi(\underline{\mathbf{k}}') - (\rho_{\underline{\mathbf{k}}} \phi)(\underline{\mathbf{k}}')| (1 + |\underline{\mathbf{k}}|)^N d\underline{\mathbf{k}}' \\ & \leq b' \int_{|\underline{\mathbf{k}}'| \geq s|\underline{\mathbf{k}}|} |\phi(\underline{\mathbf{k}}')| (1 + |\underline{\mathbf{k}}|)^N d\underline{\mathbf{k}}' \\ & \leq b'' \int_{|\underline{\mathbf{k}}'| \geq s|\underline{\mathbf{k}}|} \frac{1}{(1 + |\underline{\mathbf{k}}'|)^{M+N}} (1 + |\underline{\mathbf{k}}|)^N d\underline{\mathbf{k}}' \leq C_N; \end{aligned}$$

obviously this chain of estimates holds upon suitable choice of positive constants  $b', b'', M$  and  $C_N$ .

Now let  $\underline{\xi} = (\xi_1, \dots, \xi_{n+1}) \in R'_{n+1}$ . We distinguish two cases:

- (i)  $|\xi_1| + |\xi_{n+1}| > 0$
- (ii)  $\xi_1 = \xi_{n+1} = 0$

*Case (i).* One infers that there is some open conic neighbourhood  $E'_{\underline{\xi}}$  of  $\underline{\xi}$  with the property

$$\vartheta(|k_1| + |k_{n+1}|) \geq |\underline{\mathbf{k}}|, \quad \underline{\mathbf{k}} \in E'_{\underline{\xi}}, \quad (4.18)$$

for some suitable  $\vartheta > 0$ . Let  $\chi = \chi_1 \otimes \dots \otimes \chi_{n+1}$  with  $\chi_j \in \mathcal{S}(\mathbb{R}^d)$ ,  $j = 1, \dots, n+1$ . Recalling the notation introduced in Remark (ii) above we find

$$\widehat{\chi} * \widehat{\Psi}_{\lambda}(\underline{\mathbf{k}}) = (\widehat{\chi_1} \otimes \widehat{\chi_{n+1}}) * F_{\varphi, B_{\lambda, \underline{\mathbf{k}}}}(k_1, k_{n+1}), \quad (4.19)$$

where

$$B_{\lambda, \underline{k}} := \int e^{-i(k_2 z_2 + \dots + k_n z_n)} \chi_2(z_2) \cdots \chi_n(z_n) \cdot A_\lambda^{(1)} U(z_2) \cdots U(z_n) A_\lambda^{(n+1)} d^d z_2 \cdots d^d z_n. \quad (4.20)$$

Actually  $B_{\lambda, \underline{k}}$  is independent of  $k_1$  and  $k_{n+1}$  and  $\|B_{\lambda, \underline{k}}\| < \text{const.}$ , thus we may deduce from (4.18) together with (4.2) that

$$\begin{aligned} & \sup_{\lambda} \sup_{\underline{k} \in E'_{\underline{\xi}}} |\widehat{\chi} * \widehat{\Psi}_\lambda(\underline{k})| (1 + |\underline{k}|)^N \\ & \leq \sup_{\lambda} \sup_{\underline{k} \in E'_{\underline{\xi}}} |(\widehat{\chi}_1 \otimes \widehat{\chi}_{n+1}) * F_{\varphi, B_{\lambda, \underline{k}}}(k_1, k_{n+1})| [1 + \vartheta(|k_1| + |k_2|)]^N \leq C_N \end{aligned} \quad (4.21)$$

for all  $N \in \mathbb{N}$  with some  $C_N > 0$ .

*Case (ii).* In this case there is a conic open neighbourhood  $E''_{\underline{\xi}}$  of  $\underline{\xi}$  with  $(k_2, \dots, k_n) \in (\mathbb{R})^{n-1} \setminus \overline{V}_+ \times \cdots \times \overline{V}_+$  for all  $\underline{k} \in E''_{\underline{\xi}}$ . We may suppose that  $E''_{\underline{\xi}} = \mathbb{R}^+( \eta' \underline{\xi} + \overline{\mathcal{O}_1} )$  where  $\mathcal{O}_1$  is the unit ball around the origin in  $\mathbb{R}^{d(n+1)}$  and  $\eta'$  is some suitable number greater than 1. Now let  $\rho \in \mathcal{D}(\mathbb{R}^{d(n+1)})$ ,  $0 \leq \rho \leq 1$ , and such that  $\rho$  has support in  $\mathcal{O}_1$  and is equal to 1 on  $\frac{1}{2}\mathcal{O}_1$ . Moreover, let  $E'_{\underline{\xi}} = \mathbb{R}^+(2\eta' \underline{\xi} + \overline{\mathcal{O}_1})$ , and let  $\eta > 4\eta'$ . Then it follows that for all  $\underline{k} \in E'_{\underline{\xi}}$  and all  $\phi \in \mathcal{S}(\mathbb{R}^{d(n+1)})$  and  $\psi \in \mathcal{D}(\mathbb{R}^{d(n+1)})$  one has

$$\text{supp}(\tau_{\eta \underline{k}}(\rho_{|\underline{k}|} \phi * {}^r \psi)) \subset E''_{\underline{\xi}} \quad (4.22)$$

as soon as  $|\underline{k}|$  is large enough (depending on the support of  $\psi$ ).

Assuming now that  $\phi = \phi_1 \otimes \cdots \otimes \phi_{n+1}$  and  $\psi = \psi_1 \otimes \cdots \otimes \psi_{n+1}$  with  $\phi_j \in \mathcal{S}(\mathbb{R}^d)$  and  $\psi_j \in \mathcal{D}(\mathbb{R}^d)$ , the spectrum condition (SC) implies that, if  $\underline{k}$  is contained in  $E'_{\underline{\xi}}$  and  $|\underline{k}|$  sufficiently large, then

$$\widehat{\Psi}_\lambda(\tau_{\eta \underline{k}}(\rho_{|\underline{k}|} \phi * {}^r \psi)) = 0 \quad (4.23)$$

holds for all  $\lambda > 0$  because of (4.22) and since each  $\underline{k} \in E'_{\underline{\xi}}$  has  $(k_2, \dots, k_n) \in (\mathbb{R}^d)^{n-1} \setminus \overline{V}_+ \times \cdots \times \overline{V}_+$ . In view of (4.17) we therefore obtain that

$$\begin{aligned} & \sup_{\lambda} \sup_{\underline{k} \in E'_{\underline{\xi}}} |{}^r \phi * \psi * \widehat{\Psi}_\lambda(\eta \underline{k})| (1 + |\underline{k}|)^N \\ & \leq \sup_{\lambda} \sup_{\underline{k} \in E'_{\underline{\xi}}} |\psi * \widehat{\Psi}_\lambda(\tau_{\eta \underline{k}}(\phi - \rho_{|\underline{k}|} \phi))| (1 + |\underline{k}|)^N \\ & \quad + \sup_{\lambda} \sup_{\underline{k} \in E'_{\underline{\xi}}} |\widehat{\Psi}_\lambda(\tau_{\eta \underline{k}}(\rho_{|\underline{k}|} \phi * {}^r \psi))| (1 + |\underline{k}|)^N \\ & \leq C_N \end{aligned} \quad (4.24)$$

holds for each  $N \in \mathbb{N}$  with some suitable  $C_N > 0$ .

Now every open conic subset  $E' \subset R'_{n+1}$  with  $\overline{E'} \subset R'_{n+1}$  can be covered by finitely many conic neighbourhoods of the type  $E'_{\underline{\xi}}$ ,  $\underline{\xi} \in R'_{n+1}$ , corresponding to the cases (i)

or (ii) just considered. Relation (4.15) is thus proved by (4.21) and (4.24) apart from a remaining step which is to pass from the special functions  $\chi = \chi_1 \otimes \cdots \otimes \chi_{n+1}$  and  $\chi = \widehat{\phi_1} \widehat{r\psi_1} \otimes \cdots \otimes \widehat{\phi_{n+1}} \widehat{r\psi_{n+1}}$ , which we considered in the cases (i) and (ii), respectively, to generic  $\chi \in \mathcal{D}(\mathbb{R}^{d(n+1)})$ . The argument showing this is, however, standard [22, Lemma 8.1.1]; it is in essence contained in the proof of Prop. 2.3, and we therefore skip the details.  $\square$

The next result which we list is a simple observation combining the assumption that the Fourier-space component of the  $ACS$  is confined within a salient cone with the condition of locality, i.e. the property that elements of  $\mathcal{A}(\mathcal{O}_1)$  and  $\mathcal{A}(\mathcal{O}_2)$  commute once the localization regions are acausally separated. Then it follows that certain elements of  $\mathbb{R}^{dn} \times (\mathbb{R}^{dn} \setminus \{0\})$  are absent from  $ACS^n(\varphi)$  for hermitean functionals  $\varphi$  on  $\mathcal{A}$ . Such statements are known for the wavefront sets of Wightman distributions (they appear e.g. implicitly in [6]). Nevertheless it seems appropriate to put the simple argument on record here.

We begin by fixing the condition of locality which is motivated by Einstein causality (signals propagate with at most the velocity of light), cf. [19, 18].

(L) The theory  $(\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}), \{\alpha_x\}_{x \in \mathbb{R}^d})$  is said to fulfill *locality* if<sup>4</sup>

$$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{0\} \quad (4.25)$$

whenever the regions  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are acausally related, i.e. there is no causal curve joining  $\mathcal{O}_1$  and  $\mathcal{O}_2$  (equivalently,  $\mathcal{O}_1 \cap [\pm \overline{V}_+ + \mathcal{O}_2] = \emptyset$ ).

*Remark.* This form of locality is sometimes referred to as *spacelike commutativity*. There are theories (e.g. conformally covariant theories) which additionally fulfill *timelike commutativity* which means that (4.25) holds provided there is no timelike curve joining  $\mathcal{O}_1$  and  $\mathcal{O}_2$ . Thus, in a theory fulfilling both spacelike and timelike commutativity one has (4.25) as soon as there is no lightlike line connecting  $\mathcal{O}_1$  and  $\mathcal{O}_2$ .

We shall say that an  $n$ -tupel  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^{dn}$  is *properly acausal* (*properly non-lightlike*) if there is no causal (lightlike) curve joining any pair of points  $x_j$  and  $x_i$  for  $i \neq j$ ,  $i, j = 1, \dots, n$ . A (maximal) *salient cone*  $\mathcal{W}$  in  $\mathbb{R}^{dn} \setminus \{0\}$  is, by definition, a conic subset of  $\mathbb{R}^{dn} \setminus \{0\}$  such that  $\mathcal{W} \cap -\mathcal{W} = \emptyset$ .

With this notation, we arrive at:

**Proposition 4.2.** *Suppose that the theory  $(\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}), \{\alpha_x\}_{x \in \mathbb{R}^d})$  fulfills locality (L) and let  $\mathcal{W}$  be a closed salient cone in  $\mathbb{R}^{dn} \setminus \{0\}$ . Then for any continuous hermitean functional  $\varphi$  on  $\mathcal{A}$  the conditions*

$$ACS_{\mathbf{x}}^n(\varphi) \subset \mathcal{W} \quad \text{and} \quad \mathbf{x} \text{ properly acausal}$$

*imply*  $ACS_{\mathbf{x}}^n(\varphi) = \emptyset$ .

*The analogous statement with “properly acausal” replaced by “properly non-lightlike” holds for a theory fulfilling both spacelike and timelike commutativity.*

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<sup>4</sup>here  $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{A_1 A_2 - A_2 A_1 : A_j \in \mathcal{A}(\mathcal{O}_j), j = 1, 2\}$

**Corollary 4.3.** *For a theory satisfying locality (L) and spectrum condition (SC), it holds that*

$$ACS_{\mathbf{x}}^n(\varphi) = \emptyset$$

*if  $\varphi$  is a strongly  $C^\infty$ , continuous hermitean functional on  $\mathcal{A}$  and  $\mathbf{x}$  is properly acausal. (Again there holds the sharpened version of this statement with “ $\mathbf{x}$  properly non-lightlike” for a theory satisfying also timelike commutativity.)*

*Proof.* The Corollary follows simply from Propositions 4.1 and 4.2 since it is elementary to check that the set  $\mathcal{V}_n \setminus \{0\}$  is a closed salient cone in  $\mathbb{R}^{dn} \setminus \{0\}$ . To prove Prop. 4.2, let  $(A_\lambda^{(\mathbf{x})}) \in \mathbf{A}_{\mathbf{x}}$  for  $\mathbf{x}$  properly acausal. As a consequence of the assumptions, each  $\boldsymbol{\xi} \in \mathbb{R}^{dn} \setminus \mathcal{W}$  possesses an open neighbourhood  $V$  so that

$$\int e^{-i\lambda^{-1}\mathbf{k} \cdot \mathbf{y}} h(\mathbf{y}) \varphi(\alpha_{y_1}(A_\lambda^{(1)}) \cdots \alpha_{y_n}(A_\lambda^{(n)})) d\mathbf{y} = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad (4.26)$$

holds uniformly for  $\mathbf{k} \in V$  with some suitable  $h \in \mathcal{D}(\mathbb{R}^{dn})$ ,  $h(0) = 1$ . In view of Prop. 3.2(a) we may assume that  $h$  is real and that the diameter of the support of  $h$  is smaller than  $\frac{1}{3} \min_{i \neq j} |x_i - x_j|$ . Using Proposition 3.2(d) it follows that  $h$  and  $V$  may be chosen in such a way that one also has

$$\int e^{-i\lambda^{-1}(-\mathbf{k}) \cdot \mathbf{y}} h(\mathbf{y}) \varphi(\alpha_{y_n}(A_\lambda^{(n)}) \cdots \alpha_{y_1}(A_\lambda^{(1)})) d\mathbf{y} = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad (4.27)$$

uniformly in  $\mathbf{k} \in V$ , but since the  $\alpha_{y_j}(A_\lambda^{(j)})$ ,  $\mathbf{y} \in \text{supp } h$ ,  $j = 1, \dots, n$ , pairwise commute for sufficiently small  $\lambda$  we conclude that the left hand side of (4.26) equals the left hand side of (4.27) with  $\mathbf{k}$  replaced by  $-\mathbf{k}$ . This amounts to saying that under the stated assumptions,  $ACS_{\mathbf{x}}^n(\varphi) \subset \mathcal{W}$  entails  $ACS_{\mathbf{x}}^n(\varphi) \subset -\mathcal{W}$  and thus  $ASC_{\mathbf{x}}^n(\varphi) = \emptyset$  since  $\mathcal{W}$  is a salient cone in  $\mathbb{R}^{dn} \setminus \{0\}$ .  $\square$

In a further step we study the relation of properties of the  $ACS$  to properties of the scaling limit of the given theory in the sense of [10]. To do so we have to begin with some preparation, i.e. we need to summarize some parts of the notions developed in [10]. Let  $(\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}), \{\alpha_x\}_{x \in \mathbb{R}^d})$  be a theory as in Sec. 3, so that  $\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$  is just a net of  $C^*$ -algebras over  $d$ -dimensional Minkowski-spacetime on which the translations act as a  $C^*$ -dynamical system. We suppose such a theory is now given and keep it fixed; it will be referred to as *the given theory* or also *the underlying theory*. Note that more is not assumed presently about the given theory (like e.g. (SC), (L) or the existence of a vacuum state).

In [10], the scaling algebra associated with a given theory was introduced as a means for the analysis of the theory's short distance behaviour. The local scaling algebras at a point  $x \in \mathbb{R}^d$ , denoted by  $\underline{\mathcal{A}}_x(\mathcal{O})$ , are defined as the  $C^*$ -subalgebras of  $\mathbf{A}_x(\mathcal{O})$  formed by all the testing families  $(A_\lambda)$  with the property

$$\sup_{\lambda > 0} \|\alpha_{\lambda x}(A_\lambda) - A_\lambda\| \rightarrow 0 \quad \text{for } \lambda \rightarrow 0. \quad (4.28)$$

This property contrains the growth of the energy-momentum transferred by  $A_\lambda$  as  $\lambda \rightarrow 0$ ; see [10] for discussion. The scaling algebra at  $x$  is then defined as  $\underline{\mathcal{A}}_x :=$

$\overline{\bigcup_{\mathcal{O}} \underline{\mathcal{A}}_x(\mathcal{O})}^{C^*}$ . It is now useful to adopt the notation (cf. [10]) to write  $\underline{A}$  instead of  $(A_\lambda)$  for testing families in  $\underline{\mathcal{A}}_x$ . In other words, the function  $\underline{A} : \mathbb{R}^+ \rightarrow \mathcal{A}$  denotes the testing family  $(A_\lambda)_{\lambda>0}$  in  $\underline{\mathcal{A}}_x$ , and  $\underline{A}_\lambda$  stands for the value of that function evaluated at some argument  $\lambda \in \mathbb{R}^+$ . Then we define as in [10] the action of the translations lifted to  $\underline{\mathcal{A}}_x$  by

$$(\underline{\alpha}_y(\underline{A}))_\lambda := \alpha_{\lambda y}(\underline{A}_\lambda), \quad y \in \mathbb{R}^d, \lambda > 0, \underline{A} \in \underline{\mathcal{A}}_x. \quad (4.29)$$

One easily checks that  $\underline{\alpha}_y$  is a  $C^*$ -automorphism of the  $C^*$ -algebra  $\underline{\mathcal{A}}_x$  which acts as translation on the local scaling algebras at  $x$ , that is,

$$\underline{\alpha}_y(\underline{\mathcal{A}}_x(\mathcal{O})) = \underline{\mathcal{A}}_x(\mathcal{O} + y) \quad (4.30)$$

holds for all double cone regions  $\mathcal{O}$  and all  $x, y \in \mathbb{R}^d$ . Moreover, as a consequence of the condition (4.28) it follows that  $\{\underline{\alpha}_y\}_{y \in \mathbb{R}^d}$  is a  $C^*$ -dynamics on each  $\underline{\mathcal{A}}_x$ ,  $x \in \mathbb{R}^d$ .

Fixing some  $x \in \mathbb{R}^d$  and a state  $\omega$  on  $\underline{\mathcal{A}}_x$  one may consider the family of states  $(\omega_\lambda)_{\lambda>0}$  on  $\underline{\mathcal{A}}_x$  defined by

$$\omega_\lambda(\underline{A}) := \omega(\underline{A}_\lambda), \quad \lambda > 0, \underline{A} \in \underline{\mathcal{A}}_x, \quad (4.31)$$

as a net (generalized sequence) of states indexed by the positive reals and directed towards  $\lambda = 0$ . This net of states on the  $C^*$ -algebra  $\underline{\mathcal{A}}_x$  possesses weak-\* limit points as  $\lambda \rightarrow 0$ . The collection of these limit points is denoted by  $SL_x(\omega) = \{\omega_{0,\iota}, \iota \in \mathbb{I}_x\}$  where  $\mathbb{I}_x$  is some suitable index set labelling the collection of limit points. The states in  $SL_x(\omega)$  are called scaling limit states of  $\omega$  at  $x$ . Proceeding as in [10] one now forms the GNS-representation  $(\pi_{0,\iota}, \mathcal{H}_{0,\iota}, \Omega_{0,\iota})$  of  $\underline{\mathcal{A}}_x$  corresponding to an  $\omega_{0,\iota} \in SL_x(\omega)$ . It induces a net of  $C^*$ -algebras

$$\mathcal{O} \rightarrow \mathcal{A}_{0,\iota}(\mathcal{O}) := \pi_{0,\iota}(\underline{\mathcal{A}}_x(\mathcal{O})), \quad (4.32)$$

called the scaling limit net of the scaling limit state  $\omega_{0,\iota}$ , and provided that  $\ker \pi_{0,\iota}$  is left invariant under the action of the lifted translations  $\{\underline{\alpha}_y\}_{y \in \mathbb{R}^d}$ , there is an induced action

$$\alpha_y^{(0,\iota)}(\pi_{0,\iota}(\underline{A})) := \pi_{0,\iota}(\underline{\alpha}_y(\underline{A})), \quad (4.33)$$

$$\alpha_y^{(0,\iota)}(\mathcal{A}_{0,\iota}(\mathcal{O})) = \mathcal{A}_{0,\iota}(\mathcal{O} + y), \quad y \in \mathbb{R}^d, \underline{A} \in \underline{\mathcal{A}}_x, \quad (4.34)$$

of the translations by strongly continuous  $C^*$ -automorphisms on that scaling limit net.

Recall that a state  $\omega$  of the underlying theory is called a *vacuum state* with respect to the translation group  $\{\alpha_y\}_{y \in \mathbb{R}^d}$  if for all  $A, B \in \mathcal{A}$  the support of the Fourier-transform of  $y \mapsto \omega(A^* \alpha_y(B))$  lies in the forward lightcone  $\overline{V}_+$ . This implies that  $\omega$  is translationally invariant as a consequence of the following standard argument: Since  $\omega$  is a positive functional, it is hermitean, and so the stated constraint on the Fourier-spectrum of the action of the translations entails that the Fourier-transform of the bounded function  $x \mapsto \omega(\alpha_x(A))$  has, for all  $A = A^* \in \mathcal{A}$ , just the origin as its support. Hence it follows that the function  $x \mapsto \omega(\alpha_x(A))$  must be constant, and

by linearity, this extends to arbitrary  $A \in \mathcal{A}$ . Considering the GNS-representation  $(\pi, \mathcal{H}, \Omega)$  of  $\mathcal{A}$  corresponding to  $\omega$ , the theory  $(\mathcal{O} \rightarrow \pi(\mathcal{A}(\mathcal{O})), \{\alpha_x^\pi\}_{x \in \mathbb{R}^d})$  is then a theory fulfilling  $(SC)$ , where  $\alpha_x^\pi \circ \pi = \pi \circ \alpha_x$  is the induced action of the translations. Moreover,  $\Omega$  is a translation-invariant vacuum vector. Conversely, a theory fulfilling  $(SC)$  and possessing an invariant vacuum vector  $\Omega$  has a vacuum state  $\omega(\cdot) = \langle \Omega, \cdot \Omega \rangle$ .

It is easily proved that, if the underlying theory fulfills the locality condition  $(L)$ , then the scaling limit nets  $\mathcal{O} \rightarrow \mathcal{A}_{0,\iota}(\mathcal{O})$  corresponding to all  $\omega_{0,\iota} \in SL(\omega)$  for any state  $\omega$  on  $\mathcal{A}$  fulfill locality as well. Furthermore, if the underlying theory admits a vacuum state  $\omega$ , then one can show that each scaling limit state  $\omega_{0,\iota} \in SL_x(\omega)$  is a vacuum state on  $\underline{\mathcal{A}}_x$  with respect to the lifted translations  $\{\underline{\alpha}_y\}_{y \in \mathbb{R}^d}$  [10]. At the present level of generality, where we don't assume that the underlying theory possesses a vacuum state, we don't know if any of the scaling limit states are vacuum states on the scaling limit algebra  $\underline{\mathcal{A}}_x$ . But it turns out that certain constraints on  $ACS^2(\omega)$  for a state  $\omega$  of the underlying theory suffice to conclude that its scaling limit states are vacuum states. More precisely, we obtain the following statement.

**Theorem 4.4.** *Let  $x \in \mathbb{R}^d$ ,  $\mathbf{x} = (x, x) \in (\mathbb{R}^d)^2$ , and let  $\omega$  be any state of the underlying theory (i.e.  $\omega$  is a positive, normalized functional on  $\mathcal{A}$ ).*

(a) *Suppose that  $ACS_{\mathbf{x}}^2(\omega) \subset \mathcal{V}_2 \setminus \{0\}$ . Then each scaling limit state  $\omega_{0,\iota} \in SL_x(\omega)$  is a translationally invariant vacuum state on  $\underline{\mathcal{A}}_x$ .*

(b) *If the underlying theory satisfies the condition of locality and if  $ACS_{\mathbf{x}}^2(\omega) = \emptyset$ , then for each  $\omega_{0,\iota} \in SL_x(\omega)$  the scaling limit algebras  $\mathcal{A}_{0,\iota} = \overline{\bigcup_{\mathcal{O}} \mathcal{A}_{0,\iota}(\mathcal{O})}^{C^*}$  are Abelian.*

*Remarks.* (i) In general, the scaling limit states  $\omega_{0,\iota}$  need not be pure states on  $\underline{\mathcal{A}}_x$ . It is shown in [10] that, if the underlying theory has a pure vacuum state and fulfills locality, then the scaling limit states will be pure vacuum states for  $d \geq 3$  (but not for  $d = 2$ , cf. [11, 9])

(ii) The situation that all scaling limit algebras are Abelian is in [10] referred to by saying that  $\omega$  has a “classical scaling limit”, motivated by the fact that an Abelian algebra doesn't describe a quantum theory. In [10] it was moreover assumed that  $\omega$  is a pure vacuum state which leads for  $d \geq 3$  to the much stronger conclusion that  $\mathcal{A}_{0,\iota} = \mathbb{C}1$  for all Abelian scaling limit algebras [8].

*Proof.* (a) The statement is proved once we have shown that for any  $f \in \mathcal{S}(\mathbb{R}^d)$  whose Fourier-transform  $\widehat{f}$  has compact support in  $\mathbb{R}^d \setminus \overline{V}_+$  there holds

$$\underline{\omega}_\lambda(\underline{A}^* \underline{\alpha}_f(\underline{B})) \rightarrow 0 \quad \text{as } \lambda \rightarrow 0 \quad (4.35)$$

for all  $\underline{A}, \underline{B} \in \underline{\mathcal{A}}_x^\circ = \bigcup_{\mathcal{O}} \underline{\mathcal{A}}_x(\mathcal{O}) \subset \mathbf{A}_x$  where

$$(\underline{\alpha}_f(\underline{B}))_\lambda := \int f(y) \alpha_{\lambda y}(\underline{B}_\lambda) d^d y, \quad \lambda > 0. \quad (4.36)$$

To show this, we first use the positivity of  $\omega$  to obtain the estimate

$$\begin{aligned} |\underline{\omega}_\lambda(\underline{A}^* \underline{\alpha}_f(\underline{B}))|^2 &\leq \underline{\omega}_\lambda(\underline{A}^* \underline{A}) \underline{\omega}_\lambda(\underline{\alpha}_f(\underline{B})^* \underline{\alpha}_f(\underline{B})) \\ &\leq \|\underline{A}\|^2 \cdot \underline{\omega}_\lambda(\underline{\alpha}_f(\underline{B})^* \underline{\alpha}_f(\underline{B})). \end{aligned} \quad (4.37)$$

Furthermore, we have for each  $\lambda > 0$

$$\underline{\omega}_\lambda(\underline{\alpha}_f(\underline{B})^* \underline{\alpha}_f(\underline{B})) = \frac{1}{\lambda^{2d}} \int \overline{f(\lambda^{-1}y)} f(\lambda^{-1}y') \omega(\alpha_y(\underline{B}_\lambda^*) \alpha_{y'}(\underline{B}_\lambda)) d^d y d^d y'. \quad (4.38)$$

Now let  $U$  be an open neighbourhood of  $\text{supp } \widehat{f}$  so that  $\overline{U}$  is compact and contained in  $\mathbb{R}^d \setminus \overline{V}_+$ . Let  $\mathcal{U} := U \times -U$ , then  $\mathcal{U}$  is an open subset of  $(\mathbb{R}^d)^2 \setminus \{0\}$  such that  $\overline{\mathcal{U}}$  is compact and contained in  $(\mathbb{R}^d)^2 \setminus \mathcal{V}_2$ . Since  $ACS_{\mathbf{x}}^2(\omega) \subset \mathcal{V}_2 \setminus \{0\}$ , one can find some function  $h \in \mathcal{D}((\mathbb{R}^d)^2)$  with  $h(0) = 1$  and the property that, for all  $\underline{A} \otimes \underline{A}' \in \underline{\mathcal{A}}_x^\circ \otimes \underline{\mathcal{A}}_x^\circ$ ,

$$\int e^{-i\lambda^{-1}\mathbf{k} \cdot \mathbf{y}} h(\mathbf{y}) \omega(\alpha_y(\underline{A}_\lambda) \alpha_{y'}(\underline{A}'_\lambda)) d\mathbf{y} = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad (4.39)$$

holds uniformly for  $\mathbf{k} = (k, k') \in \mathcal{U}$  (with the obvious notation  $\mathbf{y} = (y, y')$ ). In view of Prop. 3.2(a) it may be assumed that there is a function  $h_1 \in C^\infty((\mathbb{R}^d)^2)$  which is supported outside a ball with some positive radius around the origin in  $(\mathbb{R}^d)^2$  and such that  $h + h_1 = 1$ . Since  $f$  is rapidly decaying at infinity and  $\sup_{\lambda, y, y'} |\omega(\alpha_y(\underline{B}_\lambda^*) \alpha_{y'}(\underline{B}_\lambda))| \leq \|\underline{B}\|^2$ , one obtains that

$$\frac{1}{\lambda^{2d}} \int \overline{f(\lambda^{-1}y)} f(\lambda^{-1}y') h_1(\mathbf{y}) \omega(\alpha_y(\underline{B}_\lambda^*) \alpha_{y'}(\underline{B}_\lambda)) d^d y d^d y' = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0. \quad (4.40)$$

Therefore, setting  $\omega_\lambda(\mathbf{y}) := \omega(\alpha_y(\underline{B}_\lambda^*) \alpha_{y'}(\underline{B}_\lambda))$ , the following chain of equations holds for  $\lambda \rightarrow 0$ :

$$\begin{aligned} \frac{1}{\lambda^{2d}} \int \overline{f} \otimes f(\lambda^{-1}\mathbf{y}) \omega_\lambda(\mathbf{y}) d\mathbf{y} &= \frac{1}{\lambda^{2d}} \int \overline{f} \otimes f(\lambda^{-1}\mathbf{y}) h(\mathbf{y}) \omega_\lambda(\mathbf{y}) d\mathbf{y} + O^\infty(\lambda) \\ &= \int \overline{f} \otimes f(\mathbf{y}) (h\omega_\lambda)(\lambda\mathbf{y}) d\mathbf{y} + O^\infty(\lambda) \\ &= \frac{1}{\lambda^{2d}(2\pi)^{2d}} \int \overline{\widehat{f}(k)} \widehat{f}(-k') \widehat{h\omega_\lambda}(\lambda^{-1}\mathbf{k}) d\mathbf{k} + O^\infty(\lambda) \\ &\leq \frac{1}{\lambda^{2d}(2\pi)^{2d}} \|\widehat{f}\|_{L^1}^2 \cdot \sup_{\mathbf{k} \in \mathcal{U}} \widehat{h\omega_\lambda}(\lambda^{-1}\mathbf{k}) + O^\infty(\lambda) \\ &= O^\infty(\lambda) \end{aligned} \quad (4.41)$$

where for the last estimate we have used the bound (4.39). Comparison with (4.37) and (4.38) shows that  $\underline{\omega}_\lambda(\underline{A}^* \underline{\alpha}_f(\underline{B})) = O^\infty(\lambda)$  as  $\lambda \rightarrow 0$  for all  $\underline{A}, \underline{B} \in \underline{\mathcal{A}}_x^\circ$  and  $f \in \mathcal{S}(\mathbb{R}^d)$  with  $\widehat{f}$  having compact support in  $\mathbb{R}^d \setminus \overline{V}_+$ , which yields the result.

(b) The like argument as in (a) shows that  $\underline{\omega}_\lambda(\underline{A}^* \underline{\alpha}_f(\underline{B})) = O^\infty(\lambda)$  as  $\lambda \rightarrow 0$  holds for all  $\underline{A}, \underline{B} \in \underline{\mathcal{A}}_x^\circ$  and  $f \in \mathcal{S}(\mathbb{R}^d)$  where the support of  $\widehat{f}$  is compact and doesn't contain the origin. This entails that for any choice of  $\underline{A}, \underline{B} \in \underline{\mathcal{A}}_x$  the Fourier-transform of the bounded function  $y \mapsto \omega_{0,\iota}(\underline{A}^* \underline{\alpha}_y(\underline{B}))$  has only the origin as its support and hence the function is constant. It follows that  $\alpha_y^{(0,\iota)}(B_{0,\iota}) = B_{0,\iota}$  for all  $B_{0,\iota} \in \mathcal{A}_{0,\iota}$ . Since the net  $\mathcal{O} \rightarrow \mathcal{A}_{0,\iota}(\mathcal{O})$  fulfills locality, it follows that  $B_{0,\iota}$  commutes with all elements of  $\mathcal{A}_{0,\iota}$  in view of (4.34), thus  $\mathcal{A}_{0,\iota}$  is Abelian.  $\square$

## 5 Comparison with wavefront sets of Wightman distributions

We will now specialize the setting so as to be able to compare the asymptotic correlation spectrum with the wavefront set of Wightman distributions. So we assume now that the local observable algebras  $\mathcal{A}(\mathcal{O})$  of our given theory  $(\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}), \{\alpha_x\}_{x \in \mathbb{R}^d})$  are concretely given as operator algebras on some Hilbertspace  $\mathcal{H}$ , and that the action of the translations is the adjoint action of a weakly continuous unitary group representation  $\mathbb{R}^d \ni x \mapsto U(x)$  on  $\mathcal{H}$ . To simplify the proof of Theorem 5.1 below, we will here assume that  $\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O})''$  and relax the condition that  $\{\alpha_x\}_{x \in \mathbb{R}^d}$  acts strongly continuously to the requirement of weak continuity. (However, it can be shown that the result of Thm. 5.1 also obtains when  $(\mathcal{A}, \{\alpha_x\}_{x \in \mathbb{R}^d})$  is a  $C^*$ -dynamical system with the property that all operators of the form  $\int h(x)U(x)AU(x)^{-1}dx$  are in  $\mathcal{A}(\mathcal{O})$  whenever  $A \in \mathcal{A}(\mathcal{O}_1)''$  and  $h \in \mathcal{D}(\mathcal{O}_2)$  with  $\mathcal{O}_1 + \mathcal{O}_2 \subset \mathcal{O}$ .)

Moreover, it will be assumed that there is a Wightman quantum field  $\Phi$  on  $\mathcal{H}$  (cf. [34]), i.e. a linear map  $\mathcal{D}(\mathbb{R}^d) \ni f \mapsto \Phi(f)$  which assigns to each complex-valued test-function  $f$  a closable operator  $\Phi(f)$  with a dense domain  $D_\Phi \subset \mathcal{H}$  which is independent of  $f$  and left invariant by all the  $\Phi(f)$ ; additionally, it will be supposed that  $\Phi(\bar{f}) \subset \Phi(f)^*$  where  $\bar{f}$  is the complex conjugate of  $f$  and the star denotes the adjoint operator. We also require that  $f \mapsto \Phi(f)$  is an operator-valued distribution, that is, for any  $\psi, \psi' \in D_\Phi$ , the map  $f \mapsto \langle \psi', \Phi(f)\psi \rangle$  is an element of the distribution space  $\mathcal{D}'(\mathbb{R}^d)$ . A further assumption is the covariance of the quantum field with respect to the translations of the given theory, i.e.

$$\alpha_x(\Phi(f)) = \Phi(\tau_x(f)), \quad x \in \mathbb{R}^d, f \in \mathcal{D}(\mathbb{R}^d), \quad (5.1)$$

$$U(x)D_\Phi \subset D_\Phi, \quad x \in \mathbb{R}^d. \quad (5.2)$$

Finally we assume that the quantum field is affiliated to the local von Neumann algebras of the given theory. This means that, if  $I_f|\Phi(f)|$  denotes the polar decomposition of the closed extension of  $\Phi(f)$ , then  $I_f$  and the spectral projections of  $|\Phi(f)|$  are contained in  $\mathcal{A}(\mathcal{O})''$  whenever  $\text{supp } f \subset \mathcal{O}$ . (Note that presently we make no assumptions regarding locality, spectrum condition or the existence of a vacuum state.)

The assumptions imply that for each  $\psi, \psi' \in D_\Phi$  the “ $n$ -point functionals”

$$\varphi_n(f_1 \otimes \cdots \otimes f_n) := \langle \psi', \Phi(f_1) \cdots \Phi(f_n)\psi \rangle, \quad f_j \in \mathcal{D}(\mathbb{R}^d), \quad (5.3)$$

define distributions in  $\mathcal{D}'(\mathbb{R}^{dn})$ . On the other hand, the two vectors  $\psi, \psi' \in D_\Phi$  give rise to a continuous linear functional

$$\varphi(A) := \langle \psi', A\psi \rangle, \quad A \in \mathcal{A}, \quad (5.4)$$

on the quasilocal algebra  $\mathcal{A}$ . With this notation, the following holds:

**Theorem 5.1.**  *$WF(\varphi_n) \subset ACS^n(\varphi)$  for all  $n \in \mathbb{N}$ .*

*Proof.* For all real-valued test-functions  $f$  and  $t > 0$ , the operator  $(1 + t\Phi(f)^2)^{-1}\Phi(f)$  is bounded and it holds that

$$\|(1 + t\Phi(f)^2)^{-1}\Phi(f)\| \leq t^{-1}, \quad 0 < t \leq 1. \quad (5.5)$$

Here and in the following,  $\Phi(f)^2$  is notationally identified with its Friedrich's extension. Furthermore, by the mean value theorem we have for any real  $f \in \mathcal{D}(\mathbb{R}^d)$ ,  $y \in \mathbb{R}^d$  and  $\psi \in D_\Phi$ ,

$$\|\alpha_y((1+t\Phi(f)^2)^{-1}\Phi(f) - \Phi(f))\psi\| \leq \|\Phi(\tau_y f)^3\psi\| \cdot t, \quad t > 0. \quad (5.6)$$

Now let  $n \in \mathbb{N}$ , and let  $f_1, \dots, f_n \in \mathcal{D}(\mathbb{R}^d)$  be an  $n$ -tupel of real-valued test-functions, and  $q_1, \dots, q_n$  an  $n$ -tupel of real numbers with values not less than 1. Then we write for  $j = 1, \dots, n$ :

$$f_\lambda^{(j)}(y) := f^{(j)}\left(\frac{y}{\lambda^{q_j}}\right), \quad \lambda > 0, y \in \mathbb{R}^d, \quad (5.7)$$

$$S_{p_j}(\lambda) \equiv S_{p_j}^{(j)}(\lambda) := (1 + \lambda^{p_j}\Phi(f_\lambda^{(j)})^2)^{-1}\Phi(f_\lambda^{(j)}), \quad \lambda > 0, p_j \geq 1. \quad (5.8)$$

Here we appoint the convenient convention to use the index  $j$  of  $p_j$  to distinguish the different  $S^{(j)}$ , so that the superscript  $j$  on  $S_{p_j}^{(j)}$  may be dropped without losing information.

The main step in the proof of our Theorem is to establish the following auxiliary result.

**Lemma 5.2.** *Let  $n \in \mathbb{N}$  and suppose that the  $f_j$  and  $q_j$ ,  $j = 1, \dots, n$  are given arbitrarily. Then for each  $M > 0$ , each compact subset  $K \subset \mathbb{R}^{dn}$  and each  $\psi \in D_\Phi$  one can determine numbers  $p_j \geq 1$ ,  $j = 1, \dots, n$ , such that*

$$\|\left(\alpha_{y_n}(S_{p_n}(\lambda)) \cdots \alpha_{y_1}(S_{p_1}(\lambda)) - \Phi(\tau_{y_n} f_\lambda^{(n)}) \cdots \Phi(\tau_{y_1} f_\lambda^{(1)})\right)\psi\| = O(\lambda^M) \quad \text{as } \lambda \rightarrow 0,$$

uniformly for  $(y_1, \dots, y_n) \in K$ .

*Proof.* This Lemma will be proved via induction on  $n$ , so we begin by demonstrating the statement for the case  $n = 1$ .

According to the estimate (5.9), we have

$$\|\alpha_{y_1}(S_{p_1}(\lambda) - \Phi(f_\lambda^{(1)}))\psi\| \leq \|\Phi(\tau_{y_1} f_\lambda^{(1)})^3\psi\| \cdot \lambda^{p_1}. \quad (5.9)$$

Now we make use of the fact that the  $\varphi_n$  as in (5.3) are distributions. Hence there is for the chosen  $f_1 \in \mathcal{D}(\mathbb{R}^d)$  and  $q_1 \geq 1$ , and for any compact subset  $K$  of  $\mathbb{R}^d$ , some number  $m = m(f_1, q_1, K) \geq 0$  so that

$$\sup_{y_1 \in K} \|\Phi(\tau_{y_1} f_\lambda^{(1)})^3\psi\| \leq C \cdot \lambda^{-m}, \quad 0 < \lambda \leq 1 \quad (5.10)$$

with some  $C > 0$ . Thus, when we choose for given  $M > 0$  any  $p_1 \geq M + m$ , we obtain

$$\sup_{y_1 \in K} \|\alpha_{y_1}(S_{p_1}(\lambda) - \Phi(f_\lambda^{(1)}))\psi\| \leq C \cdot \lambda^M, \quad 0 < \lambda \leq 1, \quad (5.11)$$

which proves the required statement for  $n = 1$ .

To complete the proof of the Lemma by induction, we suppose now that it holds for some arbitrary fixed  $n \in \mathbb{N}$ . We need to show that then it holds also for the next integer  $n + 1$ . So let a set  $f_j$  of real-valued test-functions and numbers  $q_j \geq 1$ ,  $j = 1, \dots, n + 1$ , be given, as well as an arbitrary  $M > 0$ . We introduce the following abbreviations:  $\mathbf{y} = (y_1, \dots, y_n)$ ,  $p = (p_1, \dots, p_n)$ ,

$$X_{p,\mathbf{y}}(\lambda) := \alpha_{y_n}(S_{p_n}(\lambda)) \cdots \alpha_{y_1}(S_{p_1}(\lambda)), \quad (5.12)$$

$$Y_{\mathbf{y}}(\lambda) := \Phi(\tau_{y_n} f_\lambda^{(n)}) \cdots \Phi(\tau_{y_1} f_\lambda^{(1)}). \quad (5.13)$$

Then it holds that

$$\begin{aligned} & \| (\alpha_{y_{n+1}}(S_{p_{n+1}}(\lambda)) X_{p,\mathbf{y}}(\lambda) - \Phi(\tau_{y_{n+1}} f_\lambda^{(n+1)}) Y_{\mathbf{y}}(\lambda)) \psi \| \\ & \leq \| \alpha_{y_{n+1}}(S_{p_{n+1}}(\lambda)) (X_{p,\mathbf{y}}(\lambda) - Y_{\mathbf{y}}(\lambda)) \psi \| \end{aligned} \quad (5.14)$$

$$+ \| \alpha_{y_{n+1}}(S_{p_{n+1}}(\lambda) - \Phi(f_\lambda^{(n+1)})) Y_{\mathbf{y}}(\lambda) \psi \|. \quad (5.15)$$

One can now apply the same argument as given for the case  $n = 1$  to gain for the term (5.18) an estimate of the type (5.12), namely

$$\begin{aligned} & \| \alpha_{y_{n+1}}(S_{p_{n+1}}(\lambda) - \Phi(f_\lambda^{(n+1)})) Y_{\mathbf{y}}(\lambda) \psi \| \\ & \leq \| \Phi(\tau_{y_{n+1}} f_\lambda^{(n+1)})^3 Y_{\mathbf{y}}(\lambda) \psi \| \cdot \lambda^{p_{n+1}}. \end{aligned} \quad (5.16)$$

Again along the lines of the arguments given for the case  $n = 1$ , we use that the quantum field is an operator-valued distribution, implying that for the given  $f_j$  and  $q_j$ ,  $j = 1, \dots, n + 1$ , and for any compact subset  $K'$  of  $\mathbb{R}^{d(n+1)}$ , there is some number  $m' \geq 0$  (depending on the said data) so that

$$\sup_{(\mathbf{y}, y_{n+1}) \in K'} \| \Phi(\tau_{y_{n+1}} f_\lambda^{(n+1)})^3 Y_{\mathbf{y}}(\lambda) \psi \| \leq C' \cdot \lambda^{-m'}, \quad 0 < \lambda \leq 1, \quad (5.17)$$

holds with a suitable  $C' > 0$ . So choosing for given  $M' > 0$  a  $p_{n+1} \geq M' + m'$ , it follows that

$$\sup_{(\mathbf{y}, y_{n+1}) \in K'} \| \alpha_{y_{n+1}}(S_{p_{n+1}}(\lambda) - \Phi(f_\lambda^{(n+1)})) Y_{\mathbf{y}}(\lambda) \psi \| \leq C' \cdot \lambda^{M'}, \quad 0 < \lambda \leq 1. \quad (5.18)$$

With this choice of  $p_{n+1}$  one gets, in view of (5.5),

$$\begin{aligned} & \| \alpha_{y_{n+1}}(S_{p_{n+1}}(\lambda)) (X_{p,\mathbf{y}}(\lambda) - Y_{\mathbf{y}}(\lambda)) \psi \| \leq \| S_{p_{n+1}}(\lambda) \| \| (X_{p,\mathbf{y}}(\lambda) - Y_{\mathbf{y}}(\lambda)) \psi \| \\ & \leq \lambda^{-p_{n+1}} \cdot \| (X_{p,\mathbf{y}}(\lambda) - Y_{\mathbf{y}}(\lambda)) \psi \|, \quad 0 < \lambda \leq 1. \end{aligned} \quad (5.19)$$

However, by the induction hypothesis, the statement of the Lemma holds for the fixed  $n$ , and so we may conclude that for the given  $f_j, q_j$ ,  $j = 1, \dots, n$ , and  $M = M' + p_{n+1}$  we find numbers  $p_1, \dots, p_n \geq 1$  with the property

$$\sup_{\mathbf{y} \in K^\circ} \| (X_{p,\mathbf{y}}(\lambda) - Y_{\mathbf{y}}(\lambda)) \psi \| = O(\lambda^M) \quad (5.20)$$

where  $K^\circ$  denotes the projection of  $K'$  onto the first  $n$  entries of vectors in  $\mathbb{R}^d$ . Combining this with (5.14-19), we see the induction hypothesis that the statement of the Lemma holds for some arbitrary  $n \in \mathbb{N}$  to imply the validity of the statement for the subsequent integer  $n + 1$ . This proves the Lemma.  $\square$

Continuing the proof of the theorem, our task is now to show that  $(\mathbf{x}; \boldsymbol{\xi}) \notin ACS^n(\varphi)$  implies  $(\mathbf{x}; \boldsymbol{\xi}) \notin WF(\varphi_n)$ . So suppose  $(\mathbf{x}; \boldsymbol{\xi}) \notin ACS^n(\varphi)$ . Now let  $f \in \mathcal{D}(\mathbb{R}^d)$  with  $\widehat{f}(0) = 1$  and let, for an arbitrarily given set of numbers  $q_1, \dots, q_n \geq 1$ ,

$$f_\lambda^{(j)}(x') := \theta(\lambda) f\left(\frac{x' - x_j}{\lambda^{q_j}}\right), \quad x' \in \mathbb{R}^d, \quad \lambda > 0, \quad (5.21)$$

where  $\theta(\lambda)$  is a cut-off function,  $\theta(\lambda) = 1$  for  $0 < \lambda < 1$ ,  $\theta(\lambda) = 0$  for  $\lambda \geq 1$ . Then let  $p_j \geq 1$ ,  $j = 1, \dots, n$ , and let  $S_{p_j}(\lambda)$  be defined as in (5.8) with the  $f_\lambda^{(j)}$  of (5.21). It is now easily seen that each testing family  $(A_\lambda^{(j)})$ ,  $j = 1, \dots, n$ , defined by  $A_\lambda^{(j)} = S_{p_j}(\lambda)$ , is an element of  $\mathbf{A}_x$ . We conclude that there are an open neighbourhood  $V$  of  $\boldsymbol{\xi}$  and  $h \in \mathcal{D}(\mathbb{R}^{dn})$  with  $h(0) = 1$  such that

$$\int e^{-i\lambda^{-1}\mathbf{k} \cdot \mathbf{y}} h(\mathbf{y}) \varphi(\alpha_{y_1}(S_{p_1}(\lambda)) \cdots \alpha_{y_n}(S_{p_n}(\lambda))) d\mathbf{y} = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad (5.22)$$

uniformly in  $\mathbf{k} \in V$ . Since the  $p_1, \dots, p_n \geq 1$  are arbitrary, application of Lemma 5.2 yields that this last relation entails

$$\int e^{-i\lambda^{-1}\mathbf{k} \cdot \mathbf{y}} h(\mathbf{y}) \varphi_n(\tau_{y_1} f_\lambda^{(1)} \otimes \cdots \otimes \tau_{y_n} f_\lambda^{(n)}) d\mathbf{y} = O^\infty(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad (5.23)$$

uniformly in  $\mathbf{k} \in V$ , where the  $f_\lambda^{(j)}$  are of the form (5.21) with  $q_1, \dots, q_n \geq 1$  given arbitrarily. Therefore, comparison with the part (a)  $\Leftrightarrow$  (c) of Prop. 2.1 shows that (5.23) just expresses that  $(\mathbf{x}; \boldsymbol{\xi}) \notin WF(\varphi_n)$ .  $\square$

Finally we present a statement guaranteeing that quite generally the wavefront set of the  $2n$ -point distributions  $\varphi_{2n}(f_1, \dots, f_{2n}) = \langle \psi, \Phi(f_1), \dots, \Phi(f_{2n}) \psi \rangle$  for  $\psi$  in the domain of the Wightman field  $\Phi$  are non-empty. Following are our assumptions: We consider a theory  $(\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}), \{\alpha_x\}_{x \in \mathbb{R}^d})$  given in a concrete Hilbertspace representation on a Hilbertspace  $\mathcal{H}$  together with a quantum field  $\Phi$  satisfying the assumptions listed at the beginning of the section (so that  $\Phi$  is affiliated to the local von Neumann algebras). Additionally, we suppose that the theory fulfills locality ( $L$ ), spectrum condition ( $SC$ ) and also that there exists an up to a phase unique vacuum vector  $\Omega \in \mathcal{H}$  which is cyclic for the algebra  $\mathcal{A}$ . If  $\mathcal{O} \subset \mathbb{R}^d$  is any open neighbourhood of the origin in  $\mathbb{R}^d$ , we denote by  $\mathcal{O}_x := \mathcal{O} + x$  the  $\mathcal{O}$ -neighbourhood of  $x$ . With these conventions, we get:

**Proposition 5.3.** *Let  $n \in \mathbb{N}$  and  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^{dn}$ , and let  $\psi \in D_\Phi$  be a unit vector which is separating for the local von Neumann algebras of the theory. Let  $\mathcal{O}$  be some neighbourhood of the origin in  $\mathbb{R}^d$  and suppose that for each choice of  $f_j \in \mathcal{D}(\mathcal{O}_{x_j})$ ,  $j = 1, \dots, n-1$ , there is some continuous function  $K_{f_1, \dots, f_{n-1}}$  on  $\mathcal{O}_{x_n} \times \mathcal{O}_{x_n}$  with values in  $\mathbb{C}$  such that*

$$\langle \psi, \Phi(\overline{f_1}) \cdots \Phi(\overline{f_{n-1}}) \Phi(g) \Phi(h) \Phi(f_{n-1}) \cdots \Phi(f_1) \psi \rangle = \int K_{f_1, \dots, f_n}(y, y') g(y) h(y') d^d y d^d y' \quad (5.24)$$

holds for all  $g, h \in \mathcal{D}(\mathcal{O}_{x_n})$ .

Then it follows that the field operators are multiples of 1, i.e. for each  $f \in \mathcal{D}(\mathbb{R}^d)$  there is some  $c_f \in \mathbb{C}$  such that  $\Phi(f) = c_f 1$ .

*Proof.* To simplify notation we will assume that  $x_n = 0$ . Let  $\mathcal{O}'$  be another neighbourhood of the origin in  $\mathbb{R}^d$  so that  $\overline{\mathcal{O}'} \subset \mathcal{O}$ . Then one can determine some real-valued  $g \in \mathcal{D}(\mathbb{R}^d)$  so that the sequence  $g_\nu$  with  $g_\nu(y) = \nu^d g(\nu y)$ ,  $\nu \in \mathbb{N}$ , converges for  $t \rightarrow 0$  to the  $\delta$ -distribution concentrated at the origin and has, moreover, the property that  $g_\nu^{(x')} := \tau_{x'}(g_\nu) \in \mathcal{D}(\mathcal{O})$  for all  $x' \in \mathcal{O}'$ . For each choice of  $x' \in \mathcal{O}'$  and  $f_j \in \mathcal{D}(\mathcal{O}_{x_j})$ ,  $j = 1, \dots, n$ , one can then see that the sequence of vectors

$$\psi_\nu^{(x')} := (i + \Phi(g_\nu^{(x')}))\Phi(f_{n-1}) \cdots \Phi(f_1)\psi, \quad \nu \in \mathbb{N}, \quad (5.25)$$

converges strongly as  $\nu \rightarrow \infty$  to some vector which we call  $\psi_\infty^{(x')}$ .

Consider, on the other hand, the sequence of resolvent operators  $R_\nu^{(x')} := (i + \Phi(g_\nu^{(x')}))^{-1}$ . Notice that we have

$$R_\nu^{(x')} = U(x')R_\nu^{(0)}U(-x'); \quad (5.26)$$

moreover, it holds that  $R_\nu^{(0)} \in \mathcal{A}(\frac{1}{\nu}\mathcal{O}^\circ)''$  for some neighbourhood  $\mathcal{O}^\circ$  of the origin in  $\mathbb{R}^d$ . Since from our assumptions it follows that  $\bigcap_{\nu \in \mathbb{N}} \mathcal{A}(\frac{1}{\nu}\mathcal{O}^\circ)'' = \mathbb{C}1$  [38], an argument due to Roberts [31] shows that  $R_\nu^{(x')}$ ,  $\nu \in \mathbb{N}$ , possesses a subsequence which converges weakly to a multiple  $c_{x'}1$  of the unit operator for some  $c_{x'} \in \mathbb{C}$ . To ease notation, we identify  $R_\nu^{(x')}$ ,  $\nu \in \mathbb{N}$ , with this subsequence. In view of (5.26) one deduces that  $c_{x'} \equiv c$  is independent of  $x' \in \mathcal{O}'$ . Therefore, we obtain for each  $x' \in \mathcal{O}'$ , each  $\psi' \in \mathcal{H}$  and each choice of  $f_j \in \mathcal{D}(\mathcal{O}_{x_j})$ ,  $j = 1, \dots, n-1$ :

$$\langle \psi', \Phi(f_{n-1}) \cdots \Phi(f_1)\psi \rangle = \langle \psi', R_\nu^{(x')}\psi_\nu^{(x')} \rangle \rightarrow c \langle \psi', \psi_\infty^{(x')} \rangle \quad \text{as } \nu \rightarrow \infty. \quad (5.27)$$

Let us now distinguish two possibilities:  $c = 0$  and  $c \neq 0$ .

$c = 0$ : This possibility will again be subdivided according to the subsequent two cases:

$n = 1$ : In that case (5.27) modifies to

$$\langle \psi', \psi \rangle = c \langle \psi', \psi_\infty^{(x')} \rangle = 0, \quad \psi' \in \mathcal{H}, \quad (5.28)$$

and hence  $\psi = 0$ . But this is impossible since  $\psi$  was required to be separating for the local von Neumann algebras. Thus  $c = 0$  is excluded for  $n = 1$ .

$n \geq 2$ : For this case we obtain

$$\langle \psi', \Phi(f_{n-1}) \cdots \Phi(f_1)\psi \rangle = 0, \quad \psi' \in \mathcal{H}, \quad (5.29)$$

for all  $f_j \in \mathcal{D}(\mathcal{O}_{x_j})$ ,  $j = 1, \dots, n-1$ . Since  $\psi$  is separating for the local von Neumann algebras to which the field operators are affiliated, this last equation entails that

$$\Phi(f_{n-1}) \cdots \Phi(f_1)\psi = 0 \quad (5.30)$$

holds for all real-valued  $f_j \in \mathcal{D}(\mathcal{O}_{x_j})$ . To see this, note first that  $B := \Phi(f_{n-1}) \cdots \Phi(f_1)$  is closable because it has a densely defined adjoint. Then a variation of the argument leading to Lemma 5.2 shows that  $S(t) = (1 + t^{p_{n-1}}|\Phi(f_{n-1})|^2)^{-1}\Phi(f_{n-1}) \cdots (1 +$

$t^{p_1}|\Phi(f_1)|^2)^{-1}\Phi(f_1)$ ,  $t > 0$ , converges for  $t \rightarrow 0$  strongly on  $D_\Phi$  to  $B$  upon suitable choice of numbers  $p_1, \dots, p_{n-1} \geq 1$ . Since  $S(t) \in \mathcal{A}(\tilde{\mathcal{O}})''$ ,  $t > 0$ , for some large enough region  $\tilde{\mathcal{O}}$ , it follows that for any  $A \in \mathcal{A}(\tilde{\mathcal{O}})'$  one has

$$\begin{aligned} & \langle B^* \phi', A \psi'' \rangle \\ &= \lim_{t \rightarrow 0} \langle S(t)^* \psi', A \psi'' \rangle = \lim_{t \rightarrow 0} \langle \psi', S(t) A \psi'' \rangle \\ &= \lim_{t \rightarrow 0} \langle \psi', A S(t) \psi'' \rangle = \langle A^* \psi', B \psi'' \rangle \end{aligned} \quad (5.31)$$

for all  $\psi', \psi'' \in D_\Phi$  and thus  $B$  is affiliated to  $\mathcal{A}(\tilde{\mathcal{O}})''$ , cf. [13, Lemma 2.3] and also references cited there. Now  $B\psi = 0$  implies  $E|B|^2\psi = 0$  for any spectral projection of  $|B|$ , entailing  $E|B|^2 = 0$  since  $\psi$  is separating for  $\mathcal{A}(\tilde{\mathcal{O}})$ . Hence  $B = 0$ .

Eqn. (5.30) implies that we can find some neighbourhood  $\mathcal{O}^*$  of the origin in  $\mathbb{R}^d$  with the property that

$$\Phi(\tau_{x_{n-1}+a_{n-1}}(f_{n-1})) \cdots \Phi(\tau_{x_1+a_1}(f_1)) = 0 \quad (5.32)$$

holds for all  $f_j \in \mathcal{D}(\mathcal{O}^*)$  and  $a_j \in \mathcal{O}^*$ ,  $j = 1, \dots, n-1$ . Consequently, for each  $f \in \mathcal{D}(\mathcal{O}^*)$  and all  $b_1, \dots, b_{n-1} \in \mathcal{O}_1^*$ , a sufficiently small neighbourhood of  $0 \in \mathbb{R}^d$ , there holds

$$U(x_{n-1} + b_{n-1})\Phi(f)U(x_{n-2} - x_{n-1} + b_{n-2})\Phi(f) \cdots U(x_1 - x_2 + b_1)\Phi(f)\Omega = 0; \quad (5.33)$$

here the  $b_j$  are the difference variables  $b_{n-1} = a_{n-1}$ ,  $b_{n-2} = a_{n-2} - a_{n-1}$ ,  $\dots$ ,  $b_1 = a_1 - a_2$ . Due to the spectrum condition (SC), the expression on the left hand side of (5.33) is, with respect to the variables  $b_1, \dots, b_{n-1}$ , the boundary value of a function which is analytic in the tube  $(\mathbb{R}^d + iV_+)^{n-1}$  so that, as a consequence of the right hand side of (5.33), it must in fact vanish for all  $b_j \in \mathbb{R}^d$ ,  $j = 1, \dots, n-2$ . It follows that  $\Phi(f)^{n-1}\Omega = 0$ ,  $f \in \mathcal{D}(\mathcal{O}_1^*)$  and, since the vacuum vector  $\Omega$  is separating for the local von Neumann algebras, we conclude that  $\Phi(f) = 0$  for all  $f \in \mathcal{D}(\mathcal{O}_1^*)$ . By covariance under translations and linearity of the field operators in the test-functions, we see that  $\Phi(f) = 0$  holds for all  $f \in \mathcal{D}(\mathbb{R}^d)$ . This verifies the statement of our Proposition in the case  $c = 0$ ,  $n \geq 2$ .

$c \neq 0$ . In view of (5.27), this entails that

$$\psi_\infty^{(x')} = \lim_{\nu \rightarrow \infty} \Phi(g_\nu^{(x')})\Phi(f_{n-1}) \cdots \Phi(f_1)\psi = c'\Phi(f_{n-1}) \cdots \Phi(f_1)\psi \quad (5.34)$$

holds for all  $f_j \in \mathcal{D}(\mathcal{O}_{x_j})$ ,  $j = 1, \dots, n-1$ , and  $x' \in \mathcal{O}'$ , where  $c' = \frac{1-ic}{c}$ . Since  $\lim_{\nu \rightarrow \infty} \int dx' h(x')\Phi(g_\nu^{(x')}) = \Phi(h)$  holds weakly on  $D_\Phi$ , for any  $h \in \mathcal{D}(\mathbb{R}^d)$ , we obtain from (5.34) the relation

$$(\Phi(h) - c'_h 1)\Phi(f_{n-1}) \cdots \Phi(f_1)\psi = 0 \quad (5.35)$$

for all  $f_j \in \mathcal{D}(\mathcal{O}_{x_j})$ ,  $j = 1, \dots, n-1$ , and  $h \in \mathcal{D}(\mathcal{O}')$ , with  $c'_h = c' \int dx' h(x')$ . As  $\psi$  is by assumption separating for the local von Neumann algebras, it follows for real-valued test-functions that

$$(\Phi(h) - c'_h 1)\Phi(f_{n-1}) \cdots \Phi(f_1) = 0, \quad h \in \mathcal{D}(\mathcal{O}'), \quad f_j \in \mathcal{D}(\mathcal{O}_{x_j}), \quad j = 1, \dots, n-1. \quad (5.36)$$

In the case  $n = 1$  we have immediately  $\Phi(h) = c'_h 1$ . Otherwise we conclude as before that there is a neighbourhood  $\mathcal{O}^*$  of the origin in  $\mathbb{R}^d$  so that

$$(\Phi(\tau_{a_n} h) - c'_1 1) \Phi(\tau_{x_{n-1}+a_{n-1}}(h)) \cdots \Phi(\tau_{x_1+a_1}(h)) = 0, \quad a_1, \dots, a_n \in \mathcal{O}^*, \quad h \in \mathcal{D}(\mathcal{O}^*). \quad (5.37)$$

By the same analyticity argument, based on the spectrum condition (SC), which we just used in the above case, it follows that the last relation actually holds for all  $a_1, \dots, a_n \in \mathbb{R}^d$ . As a consequence, we obtain that

$$|\Phi(\tau_a h) - c'_h 1|^2 |\Phi(\bar{h})|^{2(n-1)} = 0 \quad (5.38)$$

for all  $h \in \mathcal{D}(\mathcal{O}^*)$  and all  $a \in \mathbb{R}^d$ . Now we evaluate this relation on the vacuum state and get for all  $h \in \mathcal{D}(\mathcal{O}^*)$

$$0 = \langle \Omega, |\Phi(\tau_a h) - c'_h 1|^2 |\Phi(\bar{h})|^{2(n-1)} \Omega \rangle \rightarrow \|(\Phi(h) - c'_1 1)\Omega\| \|\Phi(\bar{h})^{n-1} \Omega\| \quad (5.39)$$

as  $a$  tends to spacelike infinity because of asymptotic spacelike clustering [34]. Since the vacuum vector  $\Omega$  is separating for the local von Neumann algebras we see, as before, that  $\Phi(h) = c'_h 1$  or  $\Phi(h) = 0$  for all  $h \in \mathcal{D}(\mathcal{O}^*)$ . Using translation-covariance and linearity of the field operators, finally there results  $\Phi(h) = c'_h 1$  or  $\Phi(h) = 0$  for all  $h \in \mathcal{D}(\mathbb{R}^d)$ , thus proving the claimed statement in the case  $c \neq 0$ , and so the proof is complete.  $\square$

For a theory  $(\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}), \{a_x\}_{x \in \mathbb{R}^d})$  with affiliated quantum field  $\Phi$  satisfying the assumptions of the last Proposition we thus obtain:

**Corollary 5.4.** *If the field operators  $\Phi(f)$ ,  $f \in \mathcal{D}(\mathbb{R}^d)$ , are not all multiples of the unit operator, then for each separating vector  $\psi \in D_\Phi$  and each  $\mathbf{x} \in (\mathbb{R}^d)^n$ ,  $n \in \mathbb{N}$ , the points  $(\mathbf{x}, \bar{\mathbf{x}}) \in (\mathbb{R}^d)^{2n}$  must be contained in the singular support of the  $2n$ -point distribution*

$$\varphi_{2n}(f_1, \dots, f_{2n}) = \langle \psi, \Phi(f_1) \cdots \Phi(f_{2n}) \psi \rangle, \quad f_1, \dots, f_{2n} \in \mathcal{D}(\mathbb{R}^d). \quad (5.40)$$

## 6 Summary and outlook

We have seen that it is possible to interpret the wavefront set of a distribution as an asymptotic form of the spectrum with respect to the translation group when the distribution is asymptotically localized at a point. Motivated by this observation we have defined the notion of an asymptotic correlation spectrum of states (and linear functionals) in a generic quantum field theory in operator algebraic formulation; this notion generalizes the wavefront set in this more general setting. The properties of the asymptotic correlation spectrum which we have derived support this point of view.

However, the present work investigates the asymptotic correlation spectrum only at a preliminary stage. There are several points calling for clarification and further development. For instance, it surely is to be expected that the inclusion stated in

Thm. 5.1 is proper, owing to the fact that  $\mathbf{A}_x$  is an algebra, so the testing families can be multiplied and typically the spectrum is augmented under such multiplication. To understand this relation better, an idea would be to introduce analogues of spectral subspaces (cf. [1]) in our asymptotic correlation spectrum setting.

Since we have the hope that eventually it should be possible to use microlocal analytic methods in the structural analysis of general quantum field theories in curved spacetime, the next step is to formulate an appropriate variant of the asymptotic correlation spectrum for quantum field theory in curved spacetime in the operator algebraic framework. Furthermore, it seems necessary to give a formulation of the polarization set (the generalization of the polarization set for distributions on sections in vector bundles) [12] within the operator algebraic approach once one attempts to formulate anything like a spin-statistics relation in this general setting. A development in this direction will have to address the question of how to define the concept of “spin” in the operator algebraic approach to quantum field theory in curved spacetime, and its relation to spectral properties.

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